The Kauffman Bracket Skein as an Algebra of Observables

Author(s): Doug Bullock, Charles Frohman and Joanna Kania-Bartoszyńska

THE KAUFFMAN BRACKET SKEIN
AS AN ALGEBRA OF OBSERVABLES

DOUG BULLOCK, CHARLES FROHMAN, AND JOANNA KANIA-BARTOSZYŃSKA

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ABSTRACT. We prove that the Kauffman bracket skein algebra of a cylinder
over a surface with boundary, defined over complex numbers, is isomorphic to
the observables of an appropriate lattice gauge field theory.

1. INTRODUCTION

Lattice gauge field theory brings the representation theory of an underlying man-
ifold and its quantum invariants into the same setting. Consider the case of a
cylinder over a compact, oriented surface with boundary. A lattice model of the
surface determines an algebra of gauge invariant fields (i.e. observables). In the
classical case, based on a connected, simply connected Lie group G, observables are
the characters of the fundamental group of the lattice represented in G. Wilson
loops can be understood as traces of conjugacy classes in the fundamental group of
the lattice. For the theory based on a Drinfeld-Jimbo deformation of a simple Lie
algebra $\mathfrak{g}$, the observables are a deformation quantization of the G-characters of the
surface with respect to the standard Poisson structure [2]. In the case of $U_h(sl_2)$
this, together with the classical isomorphism [1, 6], allowed us to prove that the al-
gebra of observables is the Kauffman bracket skein algebra of the surface, completed
as an algebra over formal power series.

In this paper we return to an analytic setting in which the deformation parameter
is any complex number other than a root of unity. The analogous theorem relating
observables and the Kauffman bracket skein algebra is again true. The proof is
based on the combinatorial equivalence between the Temperley-Lieb algebra and
the quantized invariant theory of SL_2; it does not explicitly use the relationship
with surface characters.

The paper is organized as follows. Section 2 recalls definitions and associated
formulas of the Kauffman bracket skein algebra. Section 3 summarizes, for the
generic parameter, the construction of a quasi-triangular matrix model of quantum
SL_2. Section 4 outlines basic definitions and constructions of lattice gauge field
theory. Section 5 describes the correspondence between skeins and intertwiners in
the Verlinde algebra for quantum SL_2. Finally, Section 6 contains a proof of the
main theorem.

2. KAUFFMAN BRACKET SKEIN ALGEBRA

Let t be a complex number that is neither 0 nor a root of unity. Suppose that F is a compact, oriented surface with boundary and I is a closed interval. Denote by $\mathcal{L}$ the set of isotopy classes of framed links in $F \times I$, including the empty link. Let $\mathbb{C}\mathcal{L}$ be the vector space with basis $\mathcal{L}$. Define S_t to be the subspace of $\mathbb{C}\mathcal{L}$ spanned by all expressions of the form $\asymp -t \smile -t^{-1} \asymp$ and $\bigcirc + t^2 + t^{-2}$, where the framed links in each expression are identical outside the balls pictured in the diagrams.

The Kauffman bracket skein algebra $K_t(F \times I)$ is the quotient $K_t(F \times I) = \mathbb{C}\mathcal{L}/S_t$. Multiplication is given by laying one link over the other. More precisely, if α and β are in $\mathcal{L}$, isotop them so that α lies in $F \times [0, \frac{1}{2})$, and β in $F \times (\frac{1}{2}, 1]$. Then $\alpha * \beta$ is the union of these two links in $F \times [0, 1]$. Extend linearly to a product on $\mathbb{C}\mathcal{L}$. Since S_t is an ideal, the product descends, making the skein module into a skein algebra. Since the algebra structure depends on the specific product structure of $F \times I$, rather than its topological type, we use the notation $K_t(F)$.

We use the standard convention of modeling a skein in $K_t(F)$ on a framed, admissibly colored, trivalent graph. An admissible coloring is an assignment of a nonnegative integer to each edge so that the colors at each vertex form admissible triples. A triple (a, b, c) is admissible if $a \le b + c$, $b \le a + c$, $c \le a + b$ and $a + b + c$ is even. The corresponding skein in $K_t(F)$ is obtained by replacing each edge labeled with the letter m by the m-th Jones–Wenzl idempotent (see [7], or [5, p. 136]), and replacing trivalent vertices with Kauffman triads (see [5, Fig. 14.7]). If s is a trivalent spine of F, then the set of skeins carried by admissible colorings of s forms a basis $\mathcal{B}_1$ for $K_t(F)$. If F is an annulus, $\mathcal{B}_1$ consists of skeins obtained by labeling the core with a Jones–Wenzl idempotent. One may think of the core as a "trivalent" spine with one vertex, whose admissible labels are $\{(n, n, 0)\}$. The space $K_t(F)$ also has a basis $\mathcal{B}_2$ consisting of all links with simple diagrams on F, i.e. with no crossings and no trivial components.

3. REPRESENTATIONS

The details of the following can be found in [3]. Let $\mathcal{A}_t$ be the unital Hopf algebra on X, Y, K, K^{-1}, with relations:

$$KX = t^2 XK, \quad KY = t^{-2} YK,$$

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}, \quad KK^{-1} = 1.$$

Let $\underline{m}$ denote the irreducible $(m + 1)$-dimensional representation of $\mathcal{A}_t$. Fixing an ordered basis for $\underline{m}$ we define linear functionals $^m c_j^i : A_t \to \mathbb{C}$ to be the coefficient in the i-th row and j-th column in the representation $\underline{m}$. The $^m c_j^i$ form a basis for the stable subalgebra $_q SL_2$ of the Hopf algebra dual A_t^o. (Here $q = t^4$.) Define

$$\overline{\mathcal{A}}_t = \prod_{m=0}^{\infty} M_{m+1}(\mathbb{C})$$

and give it the product topology. A typical element of $\overline{\mathcal{A}}_t$ is a sequence of arbitrarily chosen matrices in which the i-th term is an $(i + 1) \times (i + 1)$ matrix.

Let $\rho_m : \mathcal{A}_t \to M_{m+1}(\mathbb{C})$ be the homomorphism corresponding to the representation $\underline{m}$. The homomorphism

$$(1) \qquad\qquad\qquad \Theta : \mathcal{A}_t \to \overline{\mathcal{A}}_t,$$

given by $\Theta(Z) = (\rho_0(Z), \rho_1(Z), \rho_2(Z), \ldots)$, is injective and its image is dense in $\overline{\mathcal{A}}_t$ (see [3]). The algebra $\overline{\mathcal{A}}_t$ is the completion of $\mathcal{A}_t$ by equivalence classes of Cauchy sequences in the weak topology from $_qSL_2$. It has the structure of a topological ribbon Hopf algebra. The projection of $\overline{\mathcal{A}}_t$ onto its $(m+1)$-st factor is an irreducible representation of $\overline{\mathcal{A}}_t$, also denoted by $\underline{m}$. Composing $^mc_j^i$ with this projection yields a function on $\overline{\mathcal{A}}_t$, also called $^mc_j^i$. Thus $_qSL_2$ is understood to lie in $(\overline{\mathcal{A}}_t)^o$.

4. LATTICE GAUGE FIELD THEORY

In this section we recall basic definitions and constructions of lattice gauge field theory. Details can be found in [2].

Let Γ be an oriented, ciliated graph; i.e. the edges are oriented and the vertices carry a linear ordering of the adjacent edges. One can think of this as instructions for building a strip and disk model of an oriented surface F having Γ as a strong deformation retract. Vertices correspond to the disks, edges to strips, and the ciliation determines how to glue the strips to the disks. The surface F is called the envelope of Γ. Define a space of connections

$$\mathbb{A}(\Gamma) = \bigotimes_{\text{edges } e} (\overline{\mathcal{A}}_t)_e,$$

and define an algebra of fields

$$C[\mathbb{A}(\Gamma)] = \bigotimes_{\text{edges } e} (_qSL_2)_e.$$

Note that fields are functions on connections in the obvious way. The connections form a coalgebra with comultiplication as defined in [2]. Multiplication of fields is the convolution product dual to comultiplication of connections. There is an action of the gauge algebra,

$$\mathcal{G}(\Gamma) = \bigotimes_{\text{vertices } v} (\overline{\mathcal{A}}_t)_v,$$

on the space of connections, and adjointly on fields. The invariant part of the gauge fields under this action is called the observables, $\mathcal{O}(\Gamma)$. The multiplication of fields restricts to make $\mathcal{O}(\Gamma)$ into an algebra, which is a deformation of the $SL_2(\mathbb{C})$-characters of $\pi_1(F)$.

Let V be a representation of the Hopf algebra $\overline{\mathcal{A}}_t$, that is, V is a finite dimensional left module over $\overline{\mathcal{A}}_t$.

The dual vector space to V carries two distinct $\overline{\mathcal{A}}_t$-module structures. When $\overline{\mathcal{A}}_t$ acts on the left, the dual module is denoted by V^*. The action is

$$Z \cdot \phi(v) = \phi(S(Z) \cdot v)$$

for any $Z \in \overline{\mathcal{A}}_t$, $v \in V$ and ϕ in the dual vector space to V. When $\overline{\mathcal{A}}_t$ acts on the right, denote the dual by V' with

$$\phi(v) \cdot Z = \phi(Z \cdot v).$$

There is an alternate description of observables in terms of "colorings" of the lattice. Assigning a representation V_e to each positively oriented edge e of the lattice Γ determines a map

$$\mathbb{A}(\Gamma) \to \bigotimes_e (V_e^* \otimes V_e).$$

This yields, at each vertex, a tensor product of representations coming from the edges adjoining that vertex taken in the order given by the ciliation. Use the representation V_e for the edges e starting at a vertex and the dual V_e^* for the edges terminating there. The resulting representation at a vertex v is denoted by V_v. Finally, choose $\phi_v \in \mathrm{Inv}(V_v')$ for each vertex v. The element $\bigotimes_v(\phi_v)$ is evaluated on a connection $\bigotimes_e x_e$ by mapping $\bigotimes_e x_e$ to $\bigotimes_e(V_e^* \otimes V_e)$ and then re-parsing to an element of $\bigotimes_v V_v$. By [2, Corollary 1] every observable is a linear combination of observables of this form.

Now assume that Γ is a trivalent lattice colored by irreducible representations. The coloring is admissible if, at each vertex, the integers corresponding to the colorings of the incident edges form an admissible triple.

Proposition 1. *Suppose that Γ is an admissibly colored trivalent lattice. For each V_v there exists a non-zero dual element invariant under the right action of $\overline{\mathcal{A}}_t$. The tensor product of these invariants over all vertices defines an observable.*

The set of such observables, one for each admissible coloring, is a basis for $\mathcal{O}(\Gamma)$.

Proof. Let $c = \{\underline{m}_e \mid e$ is an edge of $\Gamma\}$ be an admissible coloring. Note that admissibility implies a 1-dimensional invariant subspace in each V_v'. Hence there is a non-zero observable $o_c = \bigotimes_v \phi_v$. Since o_c is nonzero, there exists $X_c \in \bigotimes_v V_v$ so that $o_c(X_c) \neq 0$.

Let $P : \bigotimes_v V_v \to \bigotimes_e (\underline{m}_e^* \otimes \underline{m}_e)$ be the "parsing" map. Let ι_m be the map

$$\underline{m}^* \otimes \underline{m} \cong M_{m+1}(\mathbb{C}) \to \prod_n M_n(\mathbb{C}) = \overline{\mathcal{A}}_t,$$

where the inclusion is given by forming a sequence that has all zero matrices except for the $(m+1)$-st entry corresponding to $\underline{m}$.

Define x_c to be the connection $\left(\bigotimes_e \iota_{m_e}\right)(P(X_c))$. Clearly $o_c(x_c) \neq 0$. Since $\rho_m \circ \iota_n = 0$ unless $m = n$, x_c is annihilated by all observables constructed from colorings different than c.

From this we conclude that any set of observables constructed from distinct colorings is independent. By [2] they span.   □

There is a map

(2) $$\Phi : K_t(F) \to \mathcal{O}(\Gamma)$$

that assigns to each framed link a Wilson operator. Classically, a Wilson operator is the trace of the holonomy of a connection along some fixed loop. The construction in the quantum setting is described in [2]. In a theory based on the Drinfeld-Jimbo $U_h(sl_2)$ and its Hopf algebra dual, the observables are isomorphic to the Kauffman bracket skein algebra of the lattice envelope. Since $U_h(sl_2)$ is a complete topological algebra over $\mathbb{C}[[h]]$, it was necessary to complete the skein algebra as well. The isomorphism was inferred from the classical isomorphism, the agreement of Poisson brackets, and the h-adic completion. In Section 6 we will show directly that the map Φ is an isomorphism.

5. TEMPERLEY-LIEB THEORY

In this section we recall the correspondence between skeins and intertwiners in the category of representations $\underline{m}$ of $\overline{\mathcal{A}}_t$.

Consider a rectangle $R = I \times I$ with $2n$ distinguished points: n of them on $I \times \{0\}$ and n on $I \times \{1\}$. Take the space of blackboard framed tangles with n

arc components ending at the distinguished points. Its quotient by the Kauffman bracket skein relations is denoted by $K_t(R, n)$. This quotient has an algebra structure given by placing the bottom of a rectangle on the top of another in such a way that the distinguished points meet.

Fact 1. *The algebra $K_t(R, n)$ is isomorphic to $\mathrm{End}_{\overline{\mathcal{A}}_t}(\underline{1}^{\otimes n})$, the space of $\overline{\mathcal{A}}_t$-linear maps of $\underline{1}^{\otimes n}$ to itself.*

In the basis $\{e_{1/2}, e_{-1/2}\}$ of $\underline{1}$, the isomorphism is given by a tangle functor which makes the following assignments. A local maximum is sent to the morphism $\mu : \underline{1} \otimes \underline{1} \to \underline{0}$ defined by

$$(3) \qquad \mu(e_{1/2} \otimes e_{-1/2}) = it, \quad \mu(e_{-1/2} \otimes e_{1/2}) = -it^{-1},$$

$$\mu(e_{-1/2} \otimes e_{-1/2}) = \mu(e_{1/2} \otimes e_{1/2}) = 0.$$

A local minimum is associated to the morphism $\eta : \underline{0} \to \underline{1} \otimes \underline{1}$ given by

$$\eta(1) = it e_{1/2} \otimes e_{-1/2} - it^{-1} e_{-1/2} \otimes e_{1/2}.$$

Fact 2. *The isomorphism takes the n-th Jones-Wenzl idempotent to the intertwiner that projects $\underline{1}^{\otimes n}$ onto its highest weight invariant subspace.*

Let $\mathbb{H} = \{(x, y) \mid y \geq 0\}$ be the closed upper half plane. For any n, choose $2n$ distinguished points on the x-axis, $\{(1, 2, \ldots, 2n)\}$, and form a space of blackboard-framed tangles with n arc components ending at the distinguished points. The quotient of this space by the Kauffman bracket skein relations is denoted $K_t(\mathbb{H}, 2n)$.

Fact 3. $K_t(\mathbb{H}, 2n) \cong \mathrm{Inv}\left(((\underline{1} \otimes \underline{1})^{\otimes n})'\right).$

The isomorphism is given by the same tangle functor as for Fact 1.

An admissible triple (m, n, p) determines a skein in $K_t(\mathbb{H}, m+n+p)$ consisting of a Kauffman triad with all three legs attached to the x-axis. Fact 2 gives a canonical inclusion of $\underline{m} \otimes \underline{n} \otimes \underline{p}$ into $\underline{1}^{\otimes m} \otimes \underline{1}^{\otimes n} \otimes \underline{1}^{\otimes p} \cong (\underline{1} \otimes \underline{1})^{\otimes (m+n+p)/2}$.

Fact 4. *An admissible triple (m, n, p) — equivalently, a Kauffman triad — corresponds to a nonzero vector in the 1-dimensional space $\mathrm{Inv}((\underline{m} \otimes \underline{n} \otimes \underline{p})')$.*

6. Observables and the Kauffman bracket skein algebra

Our goal is to prove a theorem analogous to [2, Theorem 10], but replacing power series by complex numbers.

Theorem 1. *Let Γ be a lattice and let F be its envelope. Assume that $t \in \mathbb{C} \setminus \{0\}$ is not a root of unity. The algebra of observables of lattice gauge field theory on Γ based on $(\overline{\mathcal{A}}_t, {}_qSL_2)$ is isomorphic to $K_t(F)$.*

Proof. From [2] we have an algebra map from $\mathbb{C}\mathcal{L}$ to $\mathcal{O}(\Gamma)$ taking a link to the corresponding Wilson operator. As in [2, Theorem 10] this map descends to $\Phi : K_t(F) \to \mathcal{O}(\Gamma)$.

The following description of Φ is implicit in [2]. Since $\mathcal{O}(\Gamma)$ is a homeomorphism invariant of F, we can assume that the lattice Γ comes from giving orientation and ciliation to a trivalent spine γ of F. Let L be an element of the basis $\mathcal{B}_2$ of $K_t(F)$ (i.e., a link with a simple diagram). Choose an orientation of L. To compute the image of L under Φ, first perform the composition

$$(4) \qquad \bigotimes_e \overline{\mathcal{A}}_t \xrightarrow{\ \Delta\ } \bigotimes_e \overline{\mathcal{A}}_t^{\otimes n(e)} \xrightarrow{\ \otimes_e \rho_1^{\otimes n(e)}\ } \bigotimes_e (\underline{1}^* \otimes \underline{1})^{\otimes n(e)}.$$
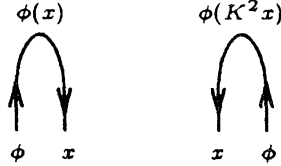
FIGURE 1. Cap tangles

Here $n(e)$ is the number of strands of L running along the edge e of Γ, and Δ means comultiplying $n(e) - 1$ times in the factor corresponding to e.

Second, in each factor where the corresponding segment of L runs against the orientation of the edge of Γ, apply $iD \otimes iD^{-1} : \underline{1}^* \otimes \underline{1} \to \underline{1} \otimes \underline{1}^*$. Here, the morphism $D : \underline{1}^* \to \underline{1}$ is defined by

$$(5) \qquad D\left(e^{1/2}\right) = ite_{-1/2}, \quad D\left(e^{-1/2}\right) = -it^{-1}e_{1/2}.$$

Third, treat each ciliated vertex as a half plane with the cilium at infinity. Up to isotopy, the link L now appears as a collection of oriented caps in each half plane. The cap pictured on the left of Figure 1 is associated with the map $\underline{1}^* \otimes \underline{1} \to \mathbb{C}$ where $\phi \otimes x \mapsto \phi(x)$, and $\underline{1} \otimes \underline{1}^* \to \mathbb{C}$ is given by $x \otimes \phi \mapsto \phi(K^2x)$ for the cap on the right.

Finally, to obtain $\Phi(L)$, multiply the result by $(-1)^{|L|}$, where $|L|$ denotes the number of components of L.

Notice that ρ_1 sends the switch map of [2] to the map $iD \otimes iD^{-1}$ and sends multiplications to contractions. Hence our description of $\Phi(L)$ for $L \in \mathcal{B}_2$ coincides with the Wilson operator. It follows from [2] that Φ does not depend on the choice of orientation of L. Extend it linearly to all of $K_t(F)$. It is a homomorphism of algebras $K_t(F)$ and $\mathcal{O}(\Gamma)$. In order to prove that Φ is an isomorphism we factor it into two maps which are isomorphisms on the level of vector spaces.

The first map, expressed in the basis $\mathcal{B}_2$, is given by a diagonal matrix with 1's and -1's on the diagonal. The second map,

$$\Phi_u : K_t(F) \to \mathcal{O}(\Gamma),$$

does not require a choice of orientation of a link L. To compute the image of $L \in \mathcal{B}_2$ under Φ_u first perform the composition (4). Second, apply the map D to each copy of $\underline{1}^*$. Third, treat each vertex as a half plane and associate the map μ from equation (3) to the (unoriented) caps.

Checking all possible orientations of L and Γ shows that $\Phi(L) = \pm\Phi_u(L)$.

By Fact 4, the map Φ_u takes an element of the basis $\mathcal{B}_1$ (i.e., a skein obtained by an admissible coloring of γ) to an observable coming from coloring the edges of Γ with corresponding irreducible representations of $\overline{\mathcal{A}_t}$. Thus, by Proposition 1, the map Φ_u takes the basis $\mathcal{B}_1$ of $K_t(F)$ to a basis of $\mathcal{O}(\Gamma)$.

As Φ and Φ_u differ by a composition with an isomorphism, both maps are isomorphisms. □

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DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, BOISE, IDAHO 83725
E-mail address: bullock@math.boisestate.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242
E-mail address: frohman@math.uiowa.edu

DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, BOISE, IDAHO 83725
E-mail address: kania@math.boisestate.edu