Integration by Substitution
and The Fundamental Theorem of Calculus
A Riemann Integral computes the signed area between the curve $y = f(x)$ and the $x$-axis on the interval $[a, b]$.

$$\text{Area} = \int_{a}^{b} f(x) \, dx$$
A Riemann Sum approximates the area by covering it with tiny rectangles, and then summing their areas.

The Riemann Integral is the limit, as the number of rectangles becomes infinite (or their width approaches 0), of the area approximations.
The Fundamental Theorem of Calculus

- The *Fundamental Theorem of Calculus* is a relationship between the area under $f(x)$ and the antiderivative of $f(x)$.

- If $f(x)$ is a continuous function on the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a)$$

- The constant, $+C$, in the antiderivative will always cancel

$$F(x) + C \bigg|_{a}^{b} = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

- This relationship is why the process of finding antiderivatives is often called the indefinite integral.
Substitution and the FTC

- The bounds (or limits) of integration, \( x = a \) and \( x = b \), are the values of \( x \) that describe the edges of the region.
- When using a \( u \)-substitution, the integrand, differential, and bounds of the transformed integral are written in terms of \( u \).
- Writing the bounds as \( x = a \) or \( u = c \) will help keep track of which values correspond to which variable.
- Notational convention is that bounds match their differential.

\[
\int_{a}^{b} f(x) \, dx = \int_{x=a}^{x=b} f(x) \, dx
\]

\[
\int_{c}^{d} g(u) \, du = \int_{u=c}^{u=d} g(u) \, du
\]
Example

Transform the following integral using $u = x^2 + 1$ then $du = 2x \, dx$

\[
\int_0^{\sqrt{2}} \frac{x}{x^2 + 1} \, dx = \int_? \frac{1}{2u} \, du
\]

The bounds need to be changed from $x$-values to $u$-values using the substitution, $u = x^2 + 1$. 
Example

Which of the following are correctly transformed integrals?

(a) \[ \int_0^{\sqrt{2}} \frac{du}{2u} \]

(b) \[ \int_{x=0}^{x=\sqrt{2}} \frac{du}{2u} \]

(c) \[ \int_1^3 \frac{du}{2u} \]

(d) \[ \int_{u=1}^{u=3} \frac{du}{2u} \]
Example

An integral and its transform have the same area.

\[
\int_0^{\sqrt{2}} \frac{x}{x^2 + 1} \, dx \quad \text{and} \quad \int_1^3 \frac{1}{2u} \, du
\]
Which of the following expressions will give the correct area?

(a) \( \frac{1}{2} \ln(u) \bigg|_0^{\sqrt{2}} \)

(b) \( \frac{1}{2} \ln(x^2 + 1) \bigg|_0^{\sqrt{2}} \)

(c) \( \frac{1}{2} \ln(u) \bigg|_1^3 \)

(d) \( \frac{1}{2} \ln(x^2 + 1) \bigg|_1^3 \)