Binomial Series
A Taylor Series of a function $f(x)$ centered at $x = c$ is

$$T(x) = \sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots$$

The coefficients of the Taylor Series are

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Coefficients can be computed either one at a time using the above formula, or they can be found using a known series.
Common Taylor Series

- **Exponential:** \(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\)

- **Sine:** \(\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\)

- **Cosine:** \(\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\)

- **Geometric:** \(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots\)

- **Binomial:** The Taylor Series for functions of the form from \((1 + x)^k\) centered at \(x = 0\). Example: \(\frac{1}{\sqrt{1+x}} = (1 + x)^{-1/2}\).
Example

Complete the worksheet to find the first few terms of the Taylor Series for \((1 + x)^{-1/2}\) centered at \(x = 0\).

\[
\begin{align*}
    a_0 &= \binom{-1/2}{0} = \frac{1}{0!} = 1 \\
    a_1 &= \binom{-1/2}{1} = \frac{-1/2}{1!} = -\frac{1}{2} \\
    a_2 &= \binom{-1/2}{2} = \frac{-1/2 \cdot -3/2}{2!} = \frac{3}{8} \\
    a_3 &= \binom{-1/2}{3} = \frac{-1/2 \cdot -3/2 \cdot -5/2}{3!} = -\frac{5}{16} \\
    a_4 &= \binom{-1/2}{4} = \frac{-1/2 \cdot -3/2 \cdot -5/2 \cdot -7/2}{4!} = \frac{35}{128}
\end{align*}
\]
The binomial coefficients are

\[
\binom{k}{n} = \frac{k(k-1)(k-2) \cdots (k-n+1)}{n!}
\]

The binomial series is

\[
(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \binom{k}{2} x^2 + \binom{k}{3} x^3 + \cdots
\]

Example:

\[
(1 + x)^{-1/2} = \binom{-1/2}{0} + \binom{-1/2}{1} x + \binom{-1/2}{2} x^2 + \binom{-1/2}{3} x^3 + \cdots
\]

\[
= 1 - \frac{1}{2} x + \frac{3}{8} x^2 - \frac{5}{16} x^3 + \cdots
\]