Math 175

Part I: Vocabulary and Notation

- The notation for limit at infinity is \( \lim_{x \to \infty} f(x) \).
  You read this: *limit of* \( f(x) \) *as* \( x \) *goes to* \( \infty \).

- There are three basic types of behavior for a function, \( f(x) \), as \( x \) approaches infinity. Know how to recognize each type and know how to write the notation for it.

1. **Horizontal Asymptote.**
   Notation is:
   \[
   \lim_{x \to \infty} f(x) = L
   \]
   where \( L \) is a number.
   We say \( f \) has a *horizontal asymptote at* \( L \).

   Examples of asymptotes:
   
   ![Horizontal Asymptote Example](image)

2. **Goes to Infinity.**
   Notation is:
   \[
   \lim_{x \to \infty} f(x) = \infty
   \]
   Or
   \[
   \lim_{x \to \infty} f(x) = -\infty
   \]

   Examples going to infinity:
   
   ![Going to Infinity Examples](image)

3. **Everything Else.**
   We say the limit *does not exist.*
   Notation is
   \[
   \lim_{x \to \infty} f(x) \text{ DNE}
   \]

   One possible DNE Example:
   
   ![DNE Example](image)

\(^1\)Or “approaches”
Part II: Speed to Infinity

- If two functions both go to infinity, it is possible that one does so faster than the other. Know how to tell which is faster, or if they are the same speed. Also know the notation:
  - If \( f \) is faster than \( g \) the notation is \( f \gg g \) or \( g \ll f \).
  - If they are the same speed the notation is \( f \sim g \).

- Be able to quickly decide the relative speeds of any of the following simple functions: logarithms, roots, powers, and exponentials. Here are the essential facts.
  - Roots are just fractional powers. Lower powers are slower than higher powers.
    \[ x^{1/2} \ll x^3 \ll x^5 \]
  - Any logarithm is slower than any power, including fractional powers.
    \[ \log_{10} x \ll \sqrt{x} \]
  - Any power is slower than any exponential that has base \( b > 1 \).
    \[ x^{100} \ll 2^x \]
  - Lower base exponentials are slower than higher base exponentials.
    \[ 2^x \ll e^x \ll 10^x \]
  - All logs are the same speed.
    \[ \log_{10} x \sim \log_2 x \]

- Be able to rank any collection of simple functions by speed. You should not have to do any algebra or calculus. Just write down the answer.

  **Example:** Rank by speed. \( x^2, 2^x, \ln x, \sqrt{x} \).

  **Solution:** \( \ln x \ll \sqrt{x} \ll x^2 \ll 2^x \)

- There is also an algebraic technique. To compare \( f \) and \( g \), assuming both diverge to infinity, compute the limit of \( f/g \).
  - If \( \lim_{x \to \infty} \left( \frac{f}{g} \right) = 0 \) then \( f \ll g \).
  - If \( \lim_{x \to \infty} \left( \frac{f}{g} \right) = \infty \) then \( f \gg g \).
  - If \( \lim_{x \to \infty} \left( \frac{f}{g} \right) = \) any non-zero number, then \( f \sim g \).
• Be able to compute \( \lim_{x \to \infty} \left( \frac{f}{g} \right) \). L'Hôpital's Rule will work. (The link includes videos.)

• You should also know how to find the limit using dominant terms of \( f \) and \( g \). If \( f \) and \( g \) have more than one term you can ignore all but the fastest term of each.

\[
\text{Example: } \lim_{x \to \infty} \frac{4x^2 - 5x + 1}{x - x^2 + 2} = \lim_{x \to \infty} \frac{4x^2}{-x^2} = -4 \quad (\text{Because the } x\text{'s cancel.})
\]

\[
\text{Example: } \lim_{x \to \infty} \frac{x^5 - 5^x}{2^x - 3^x} = \lim_{x \to \infty} \frac{-5^x}{-3^x} = \infty \quad (\text{Because } 5^x \text{ is faster than } 3^x.)
\]

Part III: Tail Thickness

• The most important examples are functions that have horizontal asymptotes at zero, as in the figure below.

![Graph of a function with a tail thickness](image)

The far right of the graph is called the tail.\(^2\)

• Be able to tell if two tails have different thickness, and if so, which is thicker.

• Tails are most commonly caused by fractions, \( \frac{1}{f(x)} \), where \( f(x) \) is any function that goes to infinity. For these functions the tail thickness rule is:

\[
\text{If } f \text{ is faster than } g, \text{ then } \frac{1}{f} \text{ has a thinner tail than } \frac{1}{g}.
\]

As with fractions, bigger denominators mean smaller fractions: 4 is bigger than 2, so \( \frac{1}{4} < \frac{1}{2} \).

• Even the notation is the same. If \( f \gg g \), then the notation for tail thickness is

\[
\frac{1}{f} \ll \frac{1}{g}
\]

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\(^2\)Sometimes it is referred to as an asymptotic tail.
• Be able to rank any collection of simple functions by tail thickness. You should not have to do any algebra or calculus. Just write down the answer, based on the speed of the denominators.

Example: Which has a thicker tail, \( \frac{1}{2x} \) or \( \frac{1}{x^3} \)?

Solution: \( \frac{1}{2x} \ll \frac{1}{x^3} \) (Because \( 2^x \) is faster than \( x^3 \).)

• Be able to replace a complicated function with a simple function that has the same tail thickness. You typically use these two facts:
  
  1. Constant multiples do not affect speed to infinity or tail thickness.
  2. You can rely on dominant terms in a quotient.

Example: Find the simplest function whose tail is the same thickness as the tail of

\[
\frac{4x^3 + 2x + 1}{3x^5 + 4x^3 + 7x}
\]

Solution:

\[
\frac{4x^3 + 2x + 1}{3x^5 + 4x^3 + 7x} \sim \frac{4x^3}{3x^5} \quad \text{(locate dominant terms)}
\]

\[
\sim \frac{x^3}{x^5} \quad \text{(ignore constant multiples)}
\]

\[
\sim \frac{1}{x^2} \quad \text{(simplify)}
\]

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3The mathematically exact method is limit comparison, as with speed to infinity. These two facts cover most limit situations. However, more complicated examples may need L’Hôpital’s rule or even more advanced limit techniques.