1. (10 pts.) The graph below shows the velocity, \( v = f(t) \), of a particle moving along a coordinate line. Explain your reasoning for each question below.

Note: Solution (b) assumes that "speeds up" means velocity increases. If instead we want speed increases, the correct answer is:

- Speeds up on \((1,2), (5,6)\)
- Slows down on \((0,1), (4,5), (6,7)\)

(a) When does the particle move forward? Backward?

Forward when \(0 \leq t \leq 1\), and when \(5 \leq t \leq 7\). Because velocity is positive.

Backward when \(1 \leq t \leq 5\). Because velocity is negative.

(b) When does the particle speed up? Slow down?

Speeds up when \(4 \leq t \leq 6\). Because velocity is increasing.

Slows down when \(0 \leq t \leq 2\) and when \(6 \leq t \leq 7\). Because vel. is decreasing.

(c) When does the particle move at its greatest speed?

\( t = 6 \). Here velocity is largest, even compared to negative velocities.

(d) When does the particle stand still for more than an instant?

Stands still when \(7 \leq t \leq 9\). Because velocity is 0.
2. (10 pts.) A road running north to south crosses a road going east to west at the point P. Car A is driving south along the first road, and car B is driving east along the second road. At a particular time car A is 10 miles to the south of P and traveling at 60 miles/hr, while car B is 15 miles to the east of P and traveling at 10 miles/hr. How fast is the distance between the two cars changing at that time?

\[ x^2 + y^2 = z^2 \]
\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \]
\[ \frac{dz}{dt} = \frac{\frac{dx}{dt} x + \frac{dy}{dt} y}{z} \]
\[ = \frac{(15)(10) + (10)(60)}{\sqrt{15^2 + 10^2}} \]
\[ = \frac{180 + 600}{\sqrt{325}} = \frac{180}{\sqrt{13}} \text{ mi/hr} \]

3. (15 pts.) Find where the curve \( y = \tan(x) + 2x \) has a horizontal tangent in the interval \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).

\[ y' = \sec^2 x + 2 = 0 \]
\[ \sec^2 x = -2 \]
\[ \text{no solution.} \]
\[ \text{no horizontal tangents.} \]
4. (15 pts.) The graph of the equation $x^2 - xy + y^2 = 16$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.

\[ x^2 = 16 \]
\[ x = \pm 4. \]

(1) (x-intercepts: set $y = 0$)

\[ 2x - 4y' - y + 2yy' = 0 \]

@ (plus in location)

@ (4, 0):
\[ 2(4) - 4y' - 0 + 0 = 0 \]
\[ \Rightarrow y' = 2 \]

@ (-4, 0):
\[ 2(-4) + 4y' - 0 + 0 = 0 \]
\[ \Rightarrow y' = 2 \]

(2) (derivative)

\[ \frac{d}{dx} (x^2 - xy + y^2) = 0 \]

\[ 2x - xy' - y + 2yy' = 0 \]

(3) Parallel because both slopes equal 2.

5. (5 pts.) Let $f(x) = \cos(3x)$. Find $dy$ if $a = \frac{\pi}{2}$ and $dx = \frac{\pi}{100}$.

\[ y' = -\sin(3x) \cdot 3 \]

\[ \Delta y = y'(\frac{\pi}{2}) \Delta x = -\sin\left(\frac{3\pi}{2}\right) \cdot 3 \cdot \frac{\pi}{100} \]

\[ = \frac{3\pi}{100} \]
6. (5 pts.) Given \( f(0) = 5 \), and \( f'(0) = 2 \) find the derivative of \( g(x) = \frac{\sin^{-1}(x) + 3x}{f(x)} \) at \( x = 0 \).

\[
g'(x) = \frac{f(x) \left[ \frac{1}{\sqrt{1-x^2}} + 3 \right] - \left[ \sin^{-1}x + 3x \right] f'(x)}{[f(x)]^2}
\]

\[
g'(0) = \frac{5 \left[ \frac{1}{\sqrt{1-0^2}} + 3 \right] - [0 + 0] \cdot 2}{5^2} = \frac{y}{5}
\]

7. (5 pts. each) Find \( y' \). Show your work and only do obvious simplifications.

(a) \( y = (x^3 + 4x)^5 \)

\[
y' = 5(x^3 + 4x)^4(3x^2 + 4)
\]

(b) \( y = (\ln(\sqrt{x}))^3 \)

\[
y' = 4(\ln x)^3 \cdot \frac{1}{x} \cdot \frac{1}{2} x^{-\frac{1}{2}}
\]

(c) \( y = xe^{\cos(x)} \)

\[
y' = 1 \cdot e^{\cos x} + x \cdot e^{\cos x} \cdot (-\sin x)
\]
8. (5 pts. each) Find \( \frac{dy}{dx} \) in each of the following equations.

(a) \( \ln(xy) + y^2 = 3x \)

\[
\frac{1}{xy} (xy' + y) + 2y y' = 3 \quad \Rightarrow \quad \frac{y'}{y} + \frac{1}{x} + 2y y' = 3 \\
\frac{y'}{y} + \frac{1}{x} + 2y y' = 3 \\
\frac{y'}{y} = \frac{3 - \frac{1}{x}}{1 + 2y} \\
y' = \left(\frac{3 - \frac{1}{x}}{1 + 2y}\right)y
\]

(b) \( y = (\cos(x))^{\sin(x)} \)

\[
\ln y = \sin x \ln(\cos x) \\
\frac{1}{y} y' = \cos x \ln(\cos x) + \sin x \cdot \frac{1}{\cos x} (-\sin x) \\
y' = (\cos x)^{\sin x} \left[ \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \right]
\]

9. (5 points) Let \( f(x) = \log_5(x^3) \). Find \( f''(x) \).

\[
f = \frac{\ln x^3}{\ln 5} = \frac{3}{\ln 5} \ln x
\]

\[
f' = \frac{3}{\ln 5} \cdot \frac{1}{x} = \frac{3}{\ln 5} x^{-1}
\]

\[
f'' = -\frac{3}{\ln 5} x^{-2}
\]
10. (10 points) Consider $y = (x^2 + 1)^{3x}$.

(a) Explain why $y' \neq 3x(x^2 + 1)^{3x-1}(2x)$.

Variable base and variable exponent

Requires logarithmic differentiation

(b) Find $y'$

$$
\ln y = 3x \ln (x^2 + 1)
$$

$$
\frac{1}{y} y' = 3 \ln (x^2 + 1) + 3x \cdot \frac{1}{x^2 + 1} \cdot 2x
$$

$$
y' = (x^2 + 1)^{3x} \left[ 3 \ln (x^2 + 1) + \frac{6x^2}{x^2 + 1} \right]
$$