3. Now try \( b = 1 \) and \( a = n/d \), a fraction where \( n \) and \( d \) have no common factor. First let \( n = 1 \) and try to determine graphically the effect of the denominator \( d \) on the shape of the graph. Then let \( n \) vary while keeping \( d \) constant. What happens when \( n = d + 1 \)?

4. What happens if \( b = 1 \) and \( a \) is irrational? Experiment with an irrational number like \( \sqrt{2} \) or \( e - 2 \). Take larger and larger values for \( \theta \) and speculate on what would happen if we were to graph the hypocycloid for all real values of \( \theta \).

5. If the circle \( C \) rolls on the outside of the fixed circle, the curve traced out by \( P \) is called an epicycloid. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

### 10.2 Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, area, arc length, and surface area.

**Tangents**

In the preceding section we saw that some curves defined by parametric equations \( x = f(t) \) and \( y = g(t) \) can also be expressed, by eliminating the parameter, in the form \( y = F(x) \). (See Exercise 67 for general conditions under which this is possible.) If we substitute \( x = f(t) \) and \( y = g(t) \) in the equation \( y = F(x) \), we get

\[
g(t) = F(f(t))
\]

and so, if \( g \), \( F \), and \( f \) are differentiable, the Chain Rule gives

\[
g'(t) = F'(f(t))f'(t) = F'(x)f'(t)
\]

If \( f'(t) \neq 0 \), we can solve for \( F'(x) \):

\[
F'(x) = \frac{g'(t)}{f'(t)}
\]

Since the slope of the tangent to the curve \( y = F(x) \) at \((x, F(x))\) is \( F'(x) \). Equation 1 enables us to find tangents to parametric curves without having to eliminate the parameter. Using Leibniz notation, we can rewrite Equation 1 in an easily remembered form:

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0
\]

It can be seen from Equation 2 that the curve has a horizontal tangent when \( \frac{dy}{dt} = 0 \) (provided that \( \frac{dx}{dt} \neq 0 \)) and it has a vertical tangent when \( \frac{dx}{dt} = 0 \) (provided that \( \frac{dy}{dt} \neq 0 \)). This information is useful for sketching parametric curves.
As we know from Chapter 4, it is also useful to consider \( d^2y/dx^2 \). This can be found by replacing \( y \) by \( dy/dx \) in Equation 2:
\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt}
\]

**EXAMPLE 1** A curve \( C \) is defined by the parametric equations \( x = t, y = t^3 - 3t \).
(a) Show that \( C \) has two tangents at the point \( (3, 0) \) and find their equations.
(b) Find the points on \( C \) where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

**SOLUTION**
(a) Notice that \( y = t^3 - 3t = t(t^2 - 3) = 0 \) when \( t = 0 \) or \( t = \pm \sqrt{3} \). Therefore the point \( (3, 0) \) on \( C \) arises from two values of the parameter, \( t = \sqrt{3} \) and \( t = -\sqrt{3} \). This indicates that \( C \) crosses itself at \( (3, 0) \). Since
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2} \left( t - \frac{1}{t} \right)
\]
the slope of the tangent when \( t = \pm \sqrt{3} \) is \( dy/dx = \pm 6/(2\sqrt{3}) = \pm \sqrt{3} \), so the equations of the tangents at \( (3, 0) \) are
\[
y = \sqrt{3} (x - 3) \quad \text{and} \quad y = -\sqrt{3} (x - 3)
\]
(b) \( C \) has a horizontal tangent when \( dy/dx = 0 \), that is, when \( dy/dt = 0 \) and \( dx/dt \neq 0 \). Since \( dy/dt = 3t^2 - 3 \), this happens when \( t^2 = 1 \), that is, \( t = \pm 1 \). The corresponding points on \( C \) are \( (1, -2) \) and \( (1, 2) \). \( C \) has a vertical tangent when \( dx/dt = 2t = 0 \), that is, \( t = 0 \). (Note that \( dy/dt \neq 0 \) there.) The corresponding point on \( C \) is \( (0, 0) \).
(c) To determine concavity we calculate the second derivative:
\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt} = \frac{3}{2} \left( 1 + \frac{1}{t^2} \right) = \frac{3(t^2 + 1)}{4t^3}
\]
Thus the curve is concave upward when \( t > 0 \) and concave downward when \( t < 0 \).
(d) Using the information from parts (b) and (c), we sketch \( C \) in Figure 1.

**EXAMPLE 2**
(a) Find the tangent to the cycloid \( x = r(\theta - \sin \theta), y = r(1 - \cos \theta) \) at the point where \( \theta = \pi/3 \). (See Example 7 in Section 10.1.)
(b) At what points is the tangent horizontal? When is it vertical?

**SOLUTION**
(a) The slope of the tangent line is
\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \sin \theta}{r(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}
\]
When $\theta = \pi/3$, we have

$$x = r \left( \frac{\pi}{3} - \sin \frac{\pi}{3} \right) = r \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \quad y = r \left( 1 - \cos \frac{\pi}{3} \right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3} \left( x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2} \right) \quad \text{or} \quad \sqrt{3}x - y = r\left( \frac{\pi}{\sqrt{3}} - 2 \right)$$

The tangent is sketched in Figure 2.

(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin \theta = 0$ and $1 - \cos \theta \neq 0$, that is, $\theta = (2n - 1)\pi$, $n$ an integer. The corresponding point on the cycloid is $((2n - 1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta \to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^+} \frac{\sin \theta}{1 - \cos \theta} = \lim_{\theta \to 2n\pi^+} \frac{\cos \theta}{\sin \theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$.

\[ \square \]

**AREAS**

We know that the area under a curve $y = F(x)$ from $a$ to $b$ is $A = \int_a^b F(x) \, dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y \, dx = \int_\alpha^\beta g(t) f'(t) \, dt \quad \text{[or]} \quad \int_\alpha^\beta g(t) f'(t) \, dt$$

**EXAMPLE 3** Find the area under one arch of the cycloid

$$x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta)$$

(See Figure 3.)
SOLUTION One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$. Using the Substitution Rule with $y = r(1 - \cos \theta)$ and $dx = r(1 - \cos \theta) \, d\theta$, we have

$$A = \int_{\theta=0}^{2\pi} r \, dy = \int_{\theta=0}^{2\pi} r(1 - \cos \theta) \, r(1 - \cos \theta) \, d\theta$$

$$= r^2 \int_{\theta=0}^{2\pi} (1 - \cos \theta)^2 \, d\theta = r^2 \int_{\theta=0}^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) \, d\theta$$

$$= r^2 \int_{\theta=0}^{2\pi} \left[ 1 - 2\cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] \, d\theta$$

$$= r^2 \left[ \frac{3}{2}\theta - 2\sin \theta + \frac{1}{2}\sin 2\theta \right]_0^{2\pi} = r^2 \left( \frac{3}{2} \cdot 2\pi \right) = 3\pi r^2$$

**ARC LENGTH**

We already know how to find the length $L$ of a curve $C$ given in the form $y = F(x)$, $a \leq x \leq b$. Formula 8.1.3 says that if $F'$ is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Suppose that $C$ can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$. This means that $C$ is traversed once, from left to right, as $t$ increases from $\alpha$ to $\beta$ and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 2 into Formula 3 and using the Substitution Rule, we obtain

$$L = \int_{\theta=0}^{\theta=\beta} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy}{dt}\right)^2 \frac{dt}{dx}} \, dx \frac{dt}{dt}$$

Since $dx/dt > 0$, we have

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Even if $C$ can't be expressed in the form $y = F(x)$, Formula 4 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into $n$ subintervals of equal width $\Delta t$. If $t_0$, $t_1$, $t_2$, ..., $t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on $C$ and the polygon with vertices $P_0, P_1, ..., P_n$ approximates $C$. (See Figure 4.)

As in Section 8.1, we define the length $L$ of $C$ to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to $f$ on the interval $[t_{i-1}, t_i]$, gives a number $t^*_i$ in $(t_{i-1}, t_i)$ such that

$$f(t_i) - f(t_{i-1}) = f(t^*_i)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t^*_i) \Delta t$$
Similarly, when applied to $g$, the Mean Value Theorem gives a number $t^{**}_i$ in $(t_{i-1}, t_i)$ such that
\[ \Delta y_i = g'(t^{**}_i) \Delta t \]
Therefore
\[ |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t^*_i)\Delta t]^2 + [g'(t^{**}_i)\Delta t]^2} \]
\[ = \sqrt{[f'(t^*_i)]^2 + [g'(t^{**}_i)]^2} \Delta t \]
and so
\[ L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t^*_i)]^2 + [g'(t^{**}_i)]^2} \Delta t \]

The sum in (5) resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t^*_i \neq t^{**}_i$ in general. Nevertheless, if $f'$ and $g'$ are continuous, it can be shown that the limit in (5) is the same as if $t^*_i$ and $t^{**}_i$ were equal, namely,
\[ L = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt \]

Thus, using Leibniz notation, we have the following result, which has the same form as Formula (4).

**Theorem** If a curve $C$ is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $f'$ and $g'$ are continuous on $[\alpha, \beta]$ and $C$ is traversed exactly once as $t$ increases from $\alpha$ to $\beta$, then the length of $C$ is
\[ L = \int_{a}^{b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

Notice that the formula in Theorem 6 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$ of Section 8.1.

**Example 4** If we use the representation of the unit circle given in Example 2 in Section 10.1,
\[ x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi \]
then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 6 gives
\[ L = \int_{0}^{2\pi} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t} \, dt = \int_{0}^{2\pi} dt = 2\pi \]
as expected. If, on the other hand, we use the representation given in Example 3 in Section 10.1,
\[ x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi \]
then $dx/dt = 2 \cos 2t$, $dy/dt = -2 \sin 2t$, and the integral in Theorem 6 gives
\[ \int_{0}^{2\pi} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{2\pi} \sqrt{4 \cos^2 2t + 4 \sin^2 2t} \, dt = \int_{0}^{2\pi} 2 \, dt = 4\pi \]
Notice that the integral gives twice the arc length of the circle because as $t$ increases from 0 to $2\pi$, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve $C$ from a parametric representation, we have to be careful to ensure that $C$ is traversed only once as $t$ increases from $\alpha$ to $\beta$.

**Example 5** Find the length of one arch of the cycloid $x = r(\theta - \sin \theta)$, $y = r(1 - \cos \theta)$.

**Solution** From Example 3 we see that one arch is described by the parameter interval $0 \leq \theta \leq 2\pi$. Since

$$
\frac{dx}{d\theta} = r(1 - \cos \theta) \quad \text{and} \quad \frac{dy}{d\theta} = r \sin \theta
$$

we have

$$
L = \int_{0}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_{0}^{2\pi} r \sqrt{(1 - \cos \theta)^2 + r^2 \sin^2 \theta} \, d\theta
$$

$$
= \int_{0}^{2\pi} r \sqrt{2 - 2 \cos \theta} \, d\theta = r \int_{0}^{2\pi} \sqrt{2(1 - \cos \theta)} \, d\theta
$$

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos \theta = 2 \sin^2(\theta/2)$. Since $0 \leq \theta \leq 2\pi$, we have $0 \leq \theta/2 \leq \pi$ and so $\sin(\theta/2) \neq 0$. Therefore

$$
\sqrt{2(1 - \cos \theta)} = \sqrt{4 \sin^2(\theta/2)} = 2|\sin(\theta/2)| = 2 \sin(\theta/2)
$$

and so

$$
L = 2r \int_{0}^{\pi} \sin(\theta/2) \, d\theta = 2r [-2 \cos(\theta/2)]_{0}^{\pi} = 8r
$$

**Surface Area**

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the $x$-axis, where $f'$, $g'$ are continuous and $g(t) \neq 0$, then the area of the resulting surface is given by

$$
S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

The general symbolic formulas $S = \int 2\pi y \, ds$ and $S = \int 2\pi x \, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

**Example 6** Show that the surface area of a sphere of radius $r$ is $4\pi r^2$.

**Solution** The sphere is obtained by rotating the semicircle

$$
x = r \cos t \quad y = r \sin t \quad 0 \leq t \leq \pi
$$
Chapter 10

Parametric Equations and Polar Coordinates

10.2 Exercises

1-2. Find \( dy/dx \).
1. \( x = t \sin t, \ y = t^2 + t \)  
2. \( x = 1/t, \ y = \sqrt{t} e^{-t} \)

3-6. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.
3. \( x = t^4 + 1, \ y = t^3 + t; \ t = -1 \)
4. \( x = t - t^{-1}; \ y = 1 + t^2; \ t = 1 \)
5. \( x = e^t, \ y = t - \ln t^2; \ t = 1 \)
6. \( x = \cos \theta + \sin 2\theta, \ y = \sin \theta + \cos 2\theta; \ \theta = 0 \)

7-8. Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.
7. \( x = 1 + \ln t, \ y = t^2 + 2; \ (1, 3) \)
8. \( x = \tan \theta, \ y = \sec \theta; \ (1, \sqrt{2}) \)

9-10. Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).
9. \( x = 6 \sin t, \ y = t^3 + t; \ (0, 0) \)
10. \( x = \cos t + \cos 2t, \ y = \sin t + \sin 2t; \ (-1, 1) \)

11-16. Find \( dy/dx \) and \( d^2y/dx^2 \). For which values of \( t \) is the curve concave upward?
11. \( x = 4 + t^2, \ y = t^3 + t^{3/2} \)
12. \( x = t^3 - 12t, \ y = t^3 - t \)
13. \( x = t - e^t, \ y = t + e^{-t} \)
14. \( x = t + \ln t, \ y = t - \ln t \)
15. \( x = 2 \sin t, \ y = 3 \cos t, \ 0 < t < 2\pi \)
16. \( x = \cos 2t, \ y = \cos t, \ 0 < t < \pi \)

17-20. Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.
17. \( x = 10 - r^2, \ y = r^3 - 12t \)
18. \( x = 2t^3 + 3t^2 - 12t, \ y = 2t^3 + 3t^2 + 1 \)
19. \( x = 2 \cos \theta, \ y = \sin 2\theta \)
20. \( x = 3t, \ y = 2 \sin \theta \)

21. Use a graph to estimate the coordinates of the rightmost point on the curve \( x = t - t^2, \ y = e^t \). Then use calculus to find the exact coordinates.
22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve \( x = t^4 - 2t^2, \ y = t + t^2 \). Then find the exact coordinates.

23-24. Graph the curve in a viewing rectangle that displays all the important aspects of the curve.
23. \( x = t^4 - 2t^2 - 2t, \ y = t^3 - t \)
24. \( x = t^4 + 4t^2 - 8t^2, \ y = 2t^3 - t \)

25. Show that the curve \( x = \cos t, \ y = \sin t \cos t \) has two tangents at \( (0, 0) \) and find their equations. Sketch the curve.
26. Graph the curve \( x = \cos t + 2 \cos 2t, \ y = \sin t + 2 \sin 2t \) to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid \( x = r \theta - r \sin \theta, \ y = r - r \cos \theta \) in terms of \( \theta \). (See Exercise 40 in Section 10.1.)
(b) Show that if \( d < r \), then the trochoid does not have a vertical tangent.
28. (a) Find the slope of the tangent to the astroid \( x = a \cos^3 \theta, y = a \sin^3 \theta \) in terms of \( \theta \). (Astroids are explored in the Laboratory Project on page 629.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?
29. At what points on the curve \( x = 2t^3, \ y = 1 - 4t - t^2 \) does the tangent line have slope 1?
30. Find equations of the tangents to the curve \( x = 3t^3 + 1, \ y = 2t^4 + 1 \) that pass through the point (4, 3).
31. Use the parametric equations of an ellipse, \( x = a \cos \theta, \ y = b \sin \theta, \ 0 \leq \theta \leq 2\pi \), to find the area that it encloses.
32. Find the area enclosed by the curve \( x = t^2 - 2t, \ y = \sqrt{t} \) and the y-axis.

33. Find the area enclosed by the x-axis and the curve \( x = 1 + e^t, \ y = t - t^2 \).

34. Find the area of the region enclosed by the astroid \( x = a \cos^3 \theta, \ y = a \sin^3 \theta \). (Astroids are explored in the Laboratory Project on page 629.)

35. Find the area under one arch of the trochoid of Exercise 40 in Section 10.1 for the case \( d < r \).

36. Let \( R \) be the region enclosed by the loop of the curve in Example 1.
   (a) Find the area of \( R \).
   (b) If \( R \) is rotated about the x-axis, find the volume of the resulting solid.
   (c) Find the centroid of \( R \).

37-40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

41. \( x = t + t^2, \ y = t^{3/2}, \ 1 \leq t \leq 2 \)

42. \( x = e^t + e^{-t}, \ y = t - 2t, \ 0 \leq t \leq 3 \)

43. \( x = \frac{t}{1+t^2}, \ y = \ln(1+t), \ 0 \leq t \leq 2 \)

44. \( x = 3 \cos t - \cos 3t, \ y = 3 \sin t - \sin 3t, \ 0 \leq t \leq \pi \)

45-47 Graph the curve and find its length.

45. \( x = e^t \cos t, \ y = e^t \sin t, \ 0 \leq t \leq \pi \)

46. \( x = \cos t + \ln(\tan \frac{t}{2}), \ y = \sin t, \ \pi/4 \leq t \leq 3\pi/4 \)

47. \( x = e^t - t, \ y = 4e^{2t}, \ -8 \leq t \leq 3 \)

48. Find the length of the loop of the curve \( x = 3t - t^3, \ y = 3t^2 \).

49. Use Simpson's Rule with \( n = 6 \) to estimate the length of the curve \( x = t^2 - e^t, \ y = t + e^t, \ -6 \leq t \leq 6 \).

50. In Exercise 43 in Section 10.1 you were asked to derive the parametric equations \( x = 2a \cot \theta, \ y = 2a \sin^2 \theta \) for the curve called the witch of Maria Agnesi. Use Simpson's Rule with \( n = 4 \) to estimate the length of the arc of this curve given by \( \pi/4 \leq \theta \leq \pi/2 \).

51-52 Find the distance traveled by a particle with position \( (x, y) \) as \( t \) varies in the given time interval. Compare with the length of the curve.

51. \( x = \sin^2 t, \ y = \cos^2 t, \ 0 \leq t \leq 3\pi \)

52. \( x = \cos^2 t, \ y = \cos t, \ 0 \leq t \leq 4\pi \)

53. Show that the total length of the ellipse \( x = a \sin \theta, \ y = b \cos \theta, \ a > b > 0 \), is

\[
L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta
\]

where \( e \) is the eccentricity of the ellipse \( e = c/a \), where \( c = \sqrt{a^2 - b^2} \).

54. Find the total length of the astroid \( x = a \cos^3 \theta, \ y = a \sin^3 \theta \), where \( a > 0 \).

55. (a) Graph the epitrochoid with equations

\[
x = 11 \cos t - 4 \cos(11t/2) \\
y = 11 \sin t - 4 \sin(11t/2)
\]

What parameter interval gives the complete curve?
(b) Use your CAS to find the approximate length of this curve.

56. A curve called Cornu's spiral is defined by the parametric equations

\[
x = C(t) = \int_0^t \cos(\pi u^2/2) \ du \\
y = S(t) = \int_0^t \sin(\pi u^2/2) \ du
\]

where \( C \) and \( S \) are the Fresnel functions that were introduced in Chapter 5.
(a) Graph this curve. What happens as \( t \to \infty \) and as \( t \to -\infty \)?
(b) Find the length of Cornu's spiral from the origin to the point with parameter value \( t \).

57-58 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

57. \( x = 1 + t e^t, \ y = (t^2 + 1) e^t, \ 0 \leq t \leq 1 \)

58. \( x = \sin^2 t, \ y = \sin 3t, \ 0 \leq t \leq \pi/3 \)
59. \( x = t^2, \ y = t^3, \ 0 \leq t \leq 1 \)
60. \( x = 3t - t^2, \ y = 3t^2, \ 0 \leq t \leq 1 \)
61. \( x = a \cos^3 \theta, \ y = a \sin^3 \theta, \ 0 \leq \theta \leq \pi/2 \)

62. Graph the curve
\[ x = 2 \cos \theta - \cos 2\theta \quad y = 2 \sin \theta - \sin 2\theta \]
If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

63. If the curve
\[ x = t + t^3 \quad y = t - \frac{1}{t} \quad 1 \leq t \leq 2 \]
is rotated about the x-axis, use your calculator to estimate the area of the resulting surface to three decimal places.

64. If the arc of the curve in Exercise 50 is rotated about the x-axis, estimate the area of the resulting surface using Simpson’s Rule with \( n = 4 \).

65-66 Find the surface area generated by rotating the given curve about the y-axis.
65. \( x = 3t^2, \ y = 2t^3, \ 0 \leq t \leq 5 \)
66. \( x = e^t - t, \ y = 4e^{t^2}, \ 0 \leq t \leq 1 \)

67. If \( f' \) is continuous and \( f''(t) \neq 0 \) for \( a \leq t \leq b \), show that the parametric curve \( x = f(t), y = g(t), a \leq t \leq b \), can be put in the form \( y = F(x) \). [Hint: Show that \( f^{-1} \) exists.]

68. Use Formula 2 to derive Formula 7 from Formula 8.2.5 for the case in which the curve can be represented in the form \( y = F(x), a \leq x \leq b \).

69. The curvature at a point \( P \) of a curve is defined as
\[ \kappa = \left| \frac{d\phi}{ds} \right| \]
where \( \phi \) is the angle of inclination of the tangent line at \( P \), as shown in the figure. Thus the curvature is the absolute value of the rate of change of \( \phi \) with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at \( P \) and will be studied in greater detail in Chapter 13.
(a) For a parametric curve \( x = x(t), y = y(t) \), derive the formula
\[ \kappa = \frac{|\dot{x} \ddot{y} - \dot{y} \ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \]
where the dots indicate derivatives with respect to \( t \), so \( \dot{x} = dx/dt \). [Hint: Use \( \phi = \tan^{-1}(\dot{y}/\dot{x}) \) and Formula 2 to find \( d\phi/dt \). Then use the Chain Rule to find \( d\phi/ds \).]

70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola \( y = x^2 \) at the point \((1, 1)\).
(b) At what point does this parabola have maximum curvature?

71. Use the formula in Exercise 69(a) to find the curvature of the cycloid \( x = t - \theta \sin \theta, y = 1 - \cos \theta \) at the top of one of its arches.

72. (a) Show that the curvature at each point of a straight line is \( \kappa = 0 \).
(b) Show that the curvature at each point of a circle of radius \( r \) is \( \kappa = 1/r \).

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point \( P \) at the end of the string is called the involute of the circle. If the circle has radius \( r \) and center \( O \) and the initial position of \( P \) is \((r, 0)\), and if the parameter \( \theta \) is chosen as in the figure, show that parametric equations of the involute are
\[ x = r(\cos \theta + \theta \sin \theta) \quad y = r(\sin \theta - \theta \cos \theta) \]

74. A cow is tied to a silo with radius \( r \) by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.