Almost everything in this lesson has a nearly identical counterpart in Lesson: Limits at Infinity. You may find it useful to have a copy of the Limits at Infinity Notes and Learning Goals handy while you read this.

Sequences

- A sequence is a discrete function that only has whole number inputs: 0, 1, 2, 3, etc.\[1\]
- The best way to understand this is to notice that the graph of a sequence is a pattern of individual dots, not a continuous curve. See examples below.
- The notation for sequences is slightly different than what you usually see for functions.
  
  Function Example: \(f(x) = x^2\).
  Sequence Example: \(a_n = n^2\).
- You can think of a sequence as an infinite list of numbers. For example, if \(a_n = n^2\), the list is 
  \[0, 1, 4, 9, 16, 25, \ldots\]
- Vocabulary: We say that \(a_n\) is the \(n\)-th term of the sequence.

Limits of Sequences

Know the notation, vocabulary, and possible behaviors for the limit of a sequence. This is almost identical to [Limits at Infinity]. The only difference is that graphs of sequences are discrete.

1. Approaches a Number.

\[
\lim_{n \to \infty} a_n = L
\]

<table>
<thead>
<tr>
<th>Examples:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lim_{n \to \infty} a_n = L)</td>
</tr>
<tr>
<td>(\lim_{n \to \infty} b_n = 0)</td>
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</tbody>
</table>

\[1\]Sometimes a sequence will start at a value other than 0.
2. Becomes Infinite.

\[ \lim_{n \to \infty} a_n = \pm \infty \]

3. Everything Else.

\[ \lim_{n \to \infty} a_n \ DNE \]
The limit does not exist.

Part II: Speed to Infinity

- Everything in [Limits at Infinity, Part II](#) is true for sequences.
- There is one new type of sequence: [factorial](#). The notation and formula for \( n \) factorial is

\[ n! = n \cdot (n - 1) \cdot (n - 2) \cdots 2 \cdot 1 \]

Examples:

\[ 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \]
\[ 1! = 1 \]
\[ 0! = 1 \quad \text{(Why 1? Same reason that anything}^0 = 1.) \]

- The speed hierarchy is the same as for functions, but with factorials added. Factorials get to infinity even faster than exponentials.

\[ \log \ll \text{roots} \ll \text{powers} \ll \text{exponentials} \ll \text{factorial} \]

Example:

\[ \log_2 n \ll n^{1/3} \ll n^5 \ll 4^n \ll n! \]
Tail Thickness

- Everything in [Limits at Infinity, Part III](#) is true for sequences.
- Factorial denominators make the thinnest tails.

**Example:** Rank the following by tail thickness: $\frac{1}{n}$, $\frac{1}{n!}$, $\frac{1}{2^n}$.

**Solution:** $\frac{1}{n!} \ll \frac{1}{2^n} \ll \frac{1}{n}$

Comparing Speed and Thickness

- Be able to use your knowledge of speed and tail thickness to compare two sequences.
- Limits of ratios are exactly as they were in [Limits at Infinity](#):
  - $\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = 0$ if and only if $a_n \ll b_n$ ($b_n$ is faster, or thicker.)
  - $\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \infty$ if and only if $a_n \gg b_n$. ($a_n$ is faster, or thicker.)
  - $\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \text{a non-zero number.}$ In this case both have the same speed or thickness.

Sequence of Partial Sums

- Suppose that $a_n$ is a sequence. For example, $a_n = n(n + 1)$:
  
  $2, 6, 12, 20, 30, \ldots$
  
  There is a new sequence called the **sequence of partial sums**. It looks like this:

  $s_1 = 2 = 2$
  $s_2 = 2 + 6 = 8$
  $s_3 = 2 + 6 + 12 = 20$
  $s_4 = 2 + 6 + 12 + 20 = 40$
  $s_5 = 2 + 6 + 12 + 20 + 30 = 70$
  
  $\vdots$
Partial sums are often written using **sigma notation**.

\[ s_n = \sum_{i=1}^{n} a_i \]

For example, \( s_5 \) from above could be written as

\[ s_5 = \sum_{i=1}^{5} a_i \text{ or } s_5 = \sum_{i=1}^{5} i(i + 1) \]

Sigma notation is particularly useful for large partial sums. Here are two ways to write \( s_{25} \).

\[ s_{25} = 2 + 6 + 12 + 20 + 30 + \cdots + 600 + 650 \]
\[ s_{25} = \sum_{i=1}^{25} i(i + 1) \]

Sigma notation is very common for symbolic partial sums. Here are two ways to write the \( n \)-th partial sum of any sequence.

\[ s_n = a_1 + a_2 + a_3 + \cdots + a_n \]
\[ s_n = \sum_{i=1}^{n} a_n \]