The previous lesson on improper integrals introduced how to compute an improper integral. The lesson also introduced the vocabulary *convergent* (finite area) or *divergent* (infinite area).

Today’s lesson will cover how to test if an improper integral is convergent or divergent by looking at the integrand. The three tests covered are:

1. Checking for a tail\(^1\) is a quick test to see if it is *at all possible* for \(\int_a^\infty f(x)\,dx\) to converge.

A function has a tail if \(\lim_{x \to \infty} f(x) = 0\). Check for a tail as follows:

- **Compute** \(\lim_{x \to \infty} f(x)\).
- **If you don’t get 0**, the function does not have a tail and there is **no chance of convergence**.
- **If you do get 0**, the function has a tail and the integral might converge. Or it might not.

**NOTE:** This is usually obvious if you draw the graph of \(f(x)\) and look at the area.

2. Be able to decide if an improper integral converges or diverges by comparing it to one you already know. There are four cases. In all cases \(f\) is the function that you already know about; \(g\) is the one you hope to learn about.

- **\(g(x) \gg f(x)\) and \(\int_0^\infty f(x)\,dx\) diverges.** \(\int_0^\infty g(x)\,dx\) must also diverge.
  
  \(g\) has a thicker tail and \(f\) creates an infinite area, so \(g\)’s tail area is infinite.

- **\(g(x) \ll f(x)\) and \(\int_0^\infty f(x)\,dx\) converges.** \(\int_0^\infty g(x)\,dx\) must also converge.
  
  \(g\) has a thinner tail and \(f\) creates a finite area, so \(g\)’s tail area is finite.

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3. Be able to judge quickly, for simple functions, if \(\int_a^\infty f(x)\,dx\) converges. Here’s how:

- **Know that convergence or divergence depends on tail thickness.** Thin tails have less area. Thick tails have more – sometimes an infinite amount.

- **Know the tail thickness hierarchy, based on** Limits at Infinity.

\[
\cdots \ll \frac{1}{e^x} \ll \frac{1}{2^x} \ll \cdots \ll \frac{1}{x^3} \ll \frac{1}{x^2} \ll \frac{1}{x} \ll \frac{1}{x^{1/2}} \ll \frac{1}{x^{1/3}} \ll \cdots \ll \frac{1}{\ln(x)} \ll \cdots
\]

\(^1\)This is known as the Divergence Test.
• Be able to correctly position any other simple function in this hierarchy.

**Example:** Where would you put \( \frac{1}{x^{1.5}} \)?

**Answer:** \( \frac{1}{x^2} \ll \frac{1}{x^{1.5}} \ll \frac{1}{x} \)

• Use the hierarchy along with at least one integral that you know. For example, this [Khan Academy video](https://www.khanacademy.org) proves that the tail of

\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx
\]

has finite area. Everything in the hierarchy to the left of \( 1/x^2 \) has a thinner tail, so also finite area.

• You will explore this in more detail in the WebAssign exercises. But by the time you take Exam 1 you should be able to immediately and correctly decide convergent or divergent for an integral involving any simple function.

4. Be able to determine if a more complicated improper integral converges or diverges. Do this by comparing it to a simple function with the same tail thickness\(^2\). Apply the method of dominant terms from [Limits at Infinity](https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/infinitydirectory/infinity.html).

**Example:** \( \int_{2}^{\infty} \frac{5x^2 + 2x + 5}{2x^4 + x^2 + 5} \, dx \). Convergent or Divergent?

**Solution:**

\[
\frac{5x^2 + 2x + 5}{2x^4 + x^2 + 5} \sim \frac{5x^2}{2x^4} \quad \text{(locate dominant terms)}
\]

\[
\sim \frac{x^2}{x^4} \quad \text{(ignore constant multiples)}
\]

\[
= \frac{1}{x^2} \quad \text{(simplify)}
\]

\( \int_{1}^{\infty} \frac{1}{x^2} \, dx \) is known convergent, so \( \int_{2}^{\infty} \frac{5x^2 + 2x + 5}{2x^4 + x^2 + 5} \, dx \) is CONVERGENT.

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\(^2\)If two integrands have the same tail thickness then both integral converge or both integrals diverge. This is called the **Limit Comparison Theorem**