Math 464, Worksheet 26

In this worksheet will add vaccination to the SIR model. As before, population of size $N$ is divided into three groups: $S$ (susceptible), $R$ (recovered/resistant), and $I$ (infectious). Assumptions are that

- Total population is constant.
- Infection rate is $\beta I/N$ per person per time step.
- Recovery rate is $\gamma$ per person per time step.
- Birth and death rates are $b$ per person per time step.
- All newborns are susceptible.
- Everyone is in exactly one group.
- From the susceptible group, there are $v$ vaccinations per person per time step.

Schematically, it looks like this:

![Schematic diagram of the SIR model with vaccinations](image)

Part I

1. Develop a model. Write your model as a system of difference equations. Eliminate $R$ using that fact that total population is a constant.

   NOTE: If you prefer numerical constants over symbolic parameters, you can use the numbers from prior SIR models:

   $$b = 0.02, \quad \beta = 0.24, \quad \gamma = 0.1, \quad \text{and} \quad N = 100$$

   However you have to keep $v$ as a symbolic parameter.

2. Find all equilibria.
3. Which equilibrium would you call “disease free”? For that equilibrium, compute the Jacobian and the eigenvalues.

4. For what values of $v$ are both eigenvalues negative?

5. Now look at the other equilibrium point (endemic). For what values of $r$ is the endemic equilibrium inside the region of positive solutions?

**Part II** Assume the disease has parameters $b = 0.03$, $\gamma = 0.07$ and $\beta = 0.4$.

6. Assume no vaccination and compute a full phase portrait. Recall that this means:
   - Analyze what happens if either $S$ or $I$ is initially zero.
   - Find all equilibria.
   - Linearize at each equilibrium.
   - For each linearization, compute eigenvalues and (if real) eigenvectors.
   - Sketch a suitable collection of possible solution curves.

7. If there is one sick person at time zero ($I(0) = 1$), what is the long term number of sick people? (Still assuming no vaccination.)

8. What if $I(0) = 100$?

9. Is there a vaccination rate for which the answer to the both of the previous questions is “Everybody gets well”?

10. If not, prove it. If so, find it and sketch the corresponding phase portrait.