Math 464, Worksheet 10

The goal in this worksheet is to compute a high power of a matrix. For example,

\[
\begin{bmatrix}
.95 & .1 \\
.1 & .89
\end{bmatrix}^{500}
\]

The process involves **eigenvalues** and **eigenvectors**, which I will define by example as they come up, and them more formally in the next worksheet.

**Part I.** The first stage is to solve the system of two equations

\[
\begin{bmatrix}
.95 - \lambda & .1 \\
.1 & .89 - \lambda
\end{bmatrix}
\begin{bmatrix}
x \\ y
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

for the three variables \(x, y\) and \(\lambda\). That’s more than 2 variables, so we don’t use normal algebra to solve this. Instead:

- Any solution with \(x = 0\) and \(y = 0\) is forbidden.
- Other solutions exist if and only if the determinant of

\[
\begin{bmatrix}
.95 - \lambda & .1 \\
.1 & .89 - \lambda
\end{bmatrix}
\]

is zero. Therefore:

1. Solve the equation

\[
(.95 - \lambda)(.89 - \lambda) - (.1)(.1) = 0
\]

Each solution is called an *eigenvalue*.

2. Pick one of the eigenvalues, say \(\lambda_1 = 1.0244\). Find one non-zero solution to the system of equations

\[
\begin{bmatrix}
.95 - 1.0244 & .1 \\
.1 & .89 - 1.0244
\end{bmatrix}
\begin{bmatrix}
x \\ y
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

There are a lot of solutions. Any one of them is called an *eigenvector associated to the eigenvalue* \(\lambda_1\)

3. Repeat Problem 2 with the other eigenvalue from Problem 1.
Part II. This is called diagonalization. It starts with a matrix made out of your eigenvectors. There are many correct ways to write this. Say you chose

\[ P = \begin{bmatrix} 1.344 & 1 \\ 1 & -1.344 \end{bmatrix} \]

1. Compute the inverse of \( P \). The rule for 2-by-2 matrices is

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{d - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

2. Check your answer by computing the matrix product \( P^{-1}P \). You should get the identity matrix. That is, a matrix with 1’s on the diagonal and 0’s elsewhere.

3. Compute

\[ P^{-1} \begin{bmatrix} .95 & .1 \\ .1 & .89 \end{bmatrix} P \]

You should get a diagonal matrix. That is, a matrix with 0’s everywhere except possibly on the diagonal. Furthermore, the diagonal entries should be your eigenvalues. Call this matrix \( D \).

4. Compute the matrix product \( PDP^{-1} \). You should get your original matrix back.

Part III. Diagonal matrices are a lot easier to compute with. Find each of the following:

1. \( D^2 \)
2. \( D^3 \)
3. \( D^{500} \)
4. \( PDP^{-1}PDP^{-1} \). Hint: matrix multiplication is associative, so you can choose your order. I recommend this progression

\[
P^{-1}P \\
(P^{-1}P)D \\
D(P^{-1}PD) \\
P(DP^{-1}PD) \\
(PDP^{-1}P)P^{-1}
\]

When you are done with this, compare to your answer on Worksheet 9, Problem 7.
5. Compute
\[
\begin{bmatrix}
.95 & .1 \\
.1 & .89
\end{bmatrix}^3
\]
Hint: write the problem as \( PDP^{-1}PDP^{-1}PDP^{-1} \) and choose a clever order for your matrix multiplications.

6. Compute
\[
\begin{bmatrix}
.95 & .1 \\
.1 & .89
\end{bmatrix}^3
\begin{bmatrix}
50 \\
50
\end{bmatrix}
\]
Compare to your answer from Worksheet 9, Problem 9.

7. Compute
\[
\begin{bmatrix}
.95 & .1 \\
.1 & .89
\end{bmatrix}^{500}
\]

8. Compute
\[
\begin{bmatrix}
.95 & .1 \\
.1 & .89
\end{bmatrix}^{500}
\begin{bmatrix}
50 \\
50
\end{bmatrix}
\]
Compare to the day 500 population from your Excel spreadsheet in Worksheet 8, Problem 2.