Eigenvalues, Eigenvectors, and Solution Curves

Assume that $J$ is the linearization of a two-by-two system at an equilibrium point.

**Case I:** $J$ has distinct, real, non-zero eigenvalues.

1. If $\lambda_1 > \lambda_2 > 0$:
   - $\lambda_1$ is the *dominant eigenvalue*.
   - $v_1$ is the *dominant eigenvector*.
   - Solutions near the equilibrium will appear approximately as shown at right.

2. If $\lambda_1 > 0\lambda_2$:
   - We rarely both with the label *dominant*.
   - Solutions near the equilibrium will appear approximately as shown at right.

3. If $0 > \lambda_1 > \lambda_2 > 0$:
   - $\lambda_1$ is the *dominant eigenvalue*. (Note that this is true even though, in absolute value, $\lambda_2$ is bigger.)
   - $v_1$ is the *dominant eigenvector*.
   - Solutions near the equilibrium will appear approximately as shown at right.
Case II: $J$ has complex eigenvalues. In this case

- The eigenvalues are complex conjugates. I.e.,
  \[ \lambda = a \pm bi \]

- Near the equilibrium solutions will either circle or spiral around the equilibrium.
- $\lambda$ will not tell you whether the solutions go clockwise or counter-clockwise.
- If $|\lambda + 1| < 1$, then solutions near the equilibrium must spiral inward.
- Computing $|\lambda + 1|$ can be a pain. You should always seek to
  1. Identify the real and imaginary parts.
  2. Compute $|\lambda + 1|^2$ and simplify if possible.

Also, if you can prove this theorem you can save yourself future computation troubles:

\[
|\lambda + 1|^2 = (a + 1)^2 + b^2, \quad \text{and} \quad |\lambda + 1|^2 = tr(J) + det(J) + 1
\]

(Recall that $tr(J)$ is the sum of the diagonal entries of $J$.)

- If $|\lambda + 1| = 1$, then solutions near the equilibrium will closely approximate closed loops, but closed loops are not guaranteed.
- If $|\lambda + 1| > 1$, then solutions near the equilibrium will spiral out.