Show all your work.

Regardless of your prior experience with calculus, you must use limit methods for all derivatives on this exam.

1. (11 pts.) The height of a moving object, measured in feet above ground, is given by

\[ h(t) = \cos(\pi t) + 2 \]

where \( t \) is time in seconds. Find the average velocity of the object on the interval from \( t = 1 \) second to \( t = 2 \) seconds.

\[
\text{ave. vel.} = \frac{h(2) - h(1)}{2 - 1} = \frac{\cos(2\pi) + 2 - (\cos\pi + 2)}{2 - 1} = \frac{1 + 2 - (1 + 2)}{1} = 2 \text{ ft/s}
\]
2. (24 pts.) Use the graph of \( f \) shown at right to answer the following questions. You may assume that the graph extends to infinity along the asymptotes.

(a) \( f(-3) = \boxed{DNE} \)?

(b) \( \lim_{x \to -3} f(x) = \frac{1}{?} \)?

(c) \( \lim_{x \to -1} f(x) = \frac{-1}{?} \)?

(d) \( \lim_{x \to 1^-} f(x) = \frac{-\infty}{?} \)?

(e) \( \lim_{x \to \infty} f(x) = \frac{-1}{?} \)?

(f) \( \lim_{x \to \pi/2} f(\sin(x + \pi/2)) = \frac{4}{?} \)?

(g) \( \lim_{x \to 2} \sin(\pi f(x)/2) = \frac{-1}{?} \)

(h) On what intervals is \( f \) continuous?

\[ (-\infty, -3), (-3, 1), (1, \infty) \]
3. (11 pts.) Given that \(-x^2 + 1 \leq \cos x \leq 1\), find

\[
\lim_{x \to 0^+} \frac{\cos x - 1}{x}
\]

\[-x^2 \leq \cos x - 1 \leq 0\]

\[-x \leq \frac{\cos x - 1}{x} \leq 0\]

\[
\lim_{x \to 0^+} x \leq \lim_{x \to 0^+} \frac{\cos x - 1}{x} \leq \lim_{x \to 0^+} 0
\]

\[
0 \leq \lim_{x \to 0^+} \frac{\cos x - 1}{x} \leq 0
\]

\[
\lim_{x \to 0^+} \frac{\cos x - 1}{x} = 0
\]
4. (11 pts.) Find the slope of the tangent line to \( f(x) = \sqrt{x^2 + 1} \) at the point \( x = 1 \).

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x^2 + 1} - 1}{x - 1} \cdot \frac{\sqrt{x^2 + 1} + 1}{\sqrt{x^2 + 1} + 1} \cdot \frac{\sqrt{x^2 + 1} + 1}{\sqrt{x^2 + 1} - 1} \cdot \frac{x^2 + 1 - 1}{(x-1)(\sqrt{x^2 + 1} + 1)}
\]

\[
= \lim_{x \to 1} \frac{x^2 + 1 - 1}{(x-1)(\sqrt{x^2 + 1} + 1)}
\]

\[
= \lim_{x \to 1} \frac{(x^2 - 1)}{(x-1)(\sqrt{x^2 + 1} + 1)}
\]

\[
= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2 + 1} + 1)}
\]

\[
= \frac{1+1}{\sqrt{1^2 + 1} + \sqrt{1}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}
\]
5. (11 pts.) Find \( f'(x) \) for \( f(x) = x^3 - 12x + 1 \).

\[
f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}
\]

\[
= \lim_{t \to x} \frac{t^2 - 12t + 1 - (x^2 - 12x + 1)}{t - x}
\]

\[
= \lim_{t \to x} \frac{t^2 - x^2 - 12t + 12x + 1 - 1}{t - x}
\]

\[
= \lim_{t \to x} \frac{(t - x)(t + x) - 12(t - x)}{(t - x)}
\]

\[
= x + x - 12
\]

\[
= 2x - 12
\]
6. (11 pts.) Where on the graph of \( f(x) = 2x^2 - 3 \) is the tangent slope \(-2\)? (x-coordinate is sufficient.)

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

\[
= \lim_{x \to a} \frac{2x^2 - 3 - (2a^2 - 3)}{x - a}
\]

\[
= \lim_{x \to a} \frac{2x^2 - 2a^2}{x - a} - 3 + 3
\]

\[
= \lim_{x \to a} \frac{2(x-a)(x+a)}{x-a}
= 2(a + a) = 4a
\]

\[
4a = -2
\]

\[
a = -\frac{1}{2}
\]
7. (11 pts.) Use the graph of $f$ shown below to sketch a graph of $f'$ on the blank axes provided.
8. (10 pts.) Two objects are falling. One was thrown upward from a 100 foot tower, so that its height (in feet) after \( t \) seconds is \( f(t) = 100 + 10t - 16t^2 \). The other was dropped, but due to air resistance its height (in feet) after \( t \) seconds is given by \( g(t) = 90 - 16t^2 + \frac{m}{4}t \). At what time (before they hit the ground) are their velocities equal?

\[ f'(t) = \lim_{x \to t} \frac{f(x) - f(t)}{x - t} = \lim_{x \to t} \frac{100 + 10x - 16x^2 - (100 + 10t - 16t^2)}{x - t} \]

\[ = \lim_{x \to t} \frac{100 - 100 + 10x - 10t - 16x^2 + 16t^2}{x - t} \]

\[ = \lim_{x \to t} \frac{10(x-t) - 16(x-t)(x+t)}{(x-t)} = 10 - 16(t+t) \]

\[ = 10 - 32t \]

\[ g'(t) = \lim_{x \to t} \frac{g(x) - g(t)}{x - t} = \lim_{x \to t} \frac{90 - 16x^2 + \frac{m}{4}x - (90 - 16t^2 + \frac{m}{4}t)}{x - t} \]

\[ = \lim_{x \to t} \frac{90 - 90 - 16x^2 + 16t^2 + \frac{m}{4}x - \frac{m}{4}t}{x - t} \]

\[ = \lim_{x \to t} \left[ \frac{-16(x-t)(x+t)}{(x-t)} + \frac{\frac{m}{4}x - \frac{m}{4}t}{x - t} \right] \]

\[ = \lim_{x \to t} \left[ \frac{-16(x-t)(x+t)}{(x-t)} + \frac{\frac{m}{4}x - \frac{m}{4}t}{x - t} \right] \]

\[ = \lim_{x \to t} \left[ -16(x+t) + \frac{40}{(x-t)} (x-t) \right] \]

\[ = -32t + \frac{40}{(4-t)^2} = 10 - 32t \]

\[ y = (4-t)^2 \]
\[ z = 4 - t \]

\[ \boxed{t = 2s} \]