1. Suppose that \( f(z) = \sum \frac{(n+1)z^n}{2^{2n+1}} \)

(a) (6 pts.) What is the domain of \( f \)?

(b) (3 pts.) Write \( f'(z) \) as a power series.

(c) (3 pts.) What is the domain of the power series for \( f'' \)?

\[
\left| \frac{n+2}{2^n+3} \cdot \frac{2^{n+1}}{n+1} \right| = \frac{n+2}{4(n+1)} \rightarrow \frac{1}{4}
\]

\[
R = 4
\]

OR

\[|z| < 4\]

b) \( f' = \sum \frac{n(n+1)z^{n-1}}{2^{n+1}} \)

OR

\[\sum \frac{(n+1)(n+2)z^n}{2^{2n+3}}\]

c) Same as \( f \) i.e. \( |z| < 4 \)
2. (10 pts.) Express \( \frac{1 + z}{(1 - z)^2} \) as a power series. Your answer must be written in the form 
\[ \sum a_n z^n, \] and you must include a description of the domain of the power series.

\[
\frac{1}{1-z} = \sum z^n = \frac{1}{(1-2)^{-1}} \quad ; \quad |z| < 1
\]

differentiate:
\[
\sum n z^{n-1} = -\frac{1}{(1-z)^2} (-1)
\]

\[
\frac{1}{(1-z)^2} = \sum (n+1) z^n \quad ; \quad |z| < 1 \quad \text{(reindexed)}
\]

partial frac:
\[
\frac{1 + z}{(1-z)^2} = \frac{A}{1 - z} + \frac{B}{(1-z)^2}
\]

\[
1 + z = A(1 - z) + B
\]
\[
= A + B - Az
\]

\[
\Rightarrow \quad A = -1, \quad A + B = 1 \Rightarrow B = 2
\]

\[
\frac{1 + z}{(1-z)^2} = -\frac{1}{1-z} + \frac{2}{1-z^2}
\]

\[
= -\sum z^n + \sum z^{n+1} z^n
\]
\[
= \sum (2n+1) z^n = \sum (z^{n+1}) z^n \quad ; \quad |z| < 1
\]

Also acceptable: \( 1 + 3z + 5z^2 + 7z^3 + \ldots \)
3. (10 pts.) Sketch the image of the region \( \{x + iy : 0 \leq x, y \leq 1 \} \) under the map \( f(z) = e^z \).

\[ e^{t+ci} = e^t \cos(c) + i e^t \sin(c) \]

\[ e^{it} = e^t \cos(t) + i e^t \sin(t) \]

\[ e^{i(0+t)} = 1 \]

\[ e^{i(1+t)} = e^{-i} \]

\[ e^{i1} = e \]

\[ e^{i0} = 1 \]
4. (6 pts.) Find all solutions of the equation $e^z = \sqrt{3} - i$.

\[ \sqrt{3} - i = 2e^{\frac{2\pi}{3}} = e^z = e^x e^{iy} \]

\[ x = \ln 2 \quad y = \frac{2\pi}{3} \]

\[ z = \ln 2 + i\left(\frac{2\pi}{3} + 2\pi n\right) \]

5. (6 pts.) Find an alpha for which $f(z) = \log_2 z^2$ is analytic at $i$ and $f(i) = 3\pi i$.

\[ i^2 - 1 = 1e^{\pi i} \]

So, any alpha not equal to $\pi + 2\pi n$ makes $f(z)$ analytic at $i$.

\[ \alpha = 2\pi \]

gives $\log_2 -1 = 3\pi i$.

Note: Any $\alpha$ in the range $\pi < \alpha < 3\pi$ would work.

6. (6 pts.) Compute the principal value of $i^{-i}$ and plot your answer on the axes at right.

\[ i^{-i} = e^{\log i (-i)} \]

\[ = e^{(0 + \frac{\pi i}{2}) (-i)} = e^{-\frac{\pi}{2}} \approx 4.8 \]

![Graphical representation of $e^{\pi i} = i$]