Math 326, Worksheet 5

For this assignment we will need some trig facts, including the definition of the complex tangent function and several real trig identities.

\[
\tan z = \frac{\sin z}{\cos z} \\
\sin 2t = 2 \sin t \cos t \\
\sinh 2t = 2 \sinh t \cosh t \\
\cos 2t = \cos^2 t - \sin^2 t \\
\cosh 2t = \cosh^2 t + \sinh^2 t \\
1 = \cos^2 t + \sin^2 t \\
1 = \cosh^2 t - \sinh^2 t
\]

1. Prove that

\[
\tan z = \frac{\sin x \cos x + i \sinh y \cosh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}
\]

2. Use identities above to prove that

\[
(\cos^2 x - \sin^2 x)(\cosh^2 y - \sinh^2 y) + (\cosh^2 y + \sinh^2 y)(\cos^2 x + \sin^2 x)
\]

simplifies to \( \cos 2x + \cosh 2y \).

3. Expand and simplify to prove that

\[
(\cos^2 x - \sin^2 x)(\cosh^2 y - \sinh^2 y) + (\cosh^2 y + \sinh^2 y)(\cos^2 x + \sin^2 x)
\]

simplifies to \( 2 \cos^2 x \cosh^2 y + 2 \sin^2 x \sinh^2 y \).

4. Notice that this proves a whopper trig identity:

\[
\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \frac{1}{2} \cos 2x + \frac{1}{2} \cosh 2y
\]

Use it to prove that

\[
\tan z = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \frac{\sinh 2y}{\cos 2x + \cosh 2y}
\]

5. Sketch the image of \( \{0 + it : t \in \mathbb{R} \} \) under \( \tan z \).

6. Sketch the image of \( \{\pi/4 + it : t \in \mathbb{R} \} \) under \( \tan z \).

7. Repeat for \( \{c + it : t \in \mathbb{R} \} \) with \( c = -\pi/4, \pm \pi/3, \pm \pi/6 \).

8. Find a fundamental domain and sketch its image.