6.6. Ordering.

6.6.1. Definition: Given \([x, y]\) and \([z, w]\) \(\in \mathbb{Q}^+\),
\[
[x, y] < [z, w] \iff x + w < z + y
\]

6.6.2. Exercise: Prove that this is well defined.

6.6.3. Definition: \(\forall X, Y \in \mathbb{Q}\)

1. \(X > Y \iff Y < X\)
2. \(X \leq Y \iff X = Y\) or \(X < Y\)
3. \(X \geq Y \iff Y \leq X\)

6.6.4. Exercise: For all \(X, Y \in \mathbb{Q}\), prove that exactly one of \(X = Y\), \(X < Y\), \(X > Y\) is true.

6.6.5. Exercises: For all \(X, Y, Z, W \in \mathbb{Q}\)

1. \(X \geq Y\) and \(Y \geq Z\) \(\implies X \geq Z\).
2. \(X > Y \iff X + Z > Y + Z\)
3. \(X \geq Y\) and \(Z \geq W\) \(\implies X + Z \geq Y + W\)
4. If \(Z > 0\), then \(X > Y \iff XZ > YZ\)
5. If \(Z < 0\), then \(X > Y \iff XZ < YZ\)
6. If \(Z \neq 0\), then \(X = Y \iff XZ = YZ\)

6.6.6. Exercise: \(\forall X, Y \in \mathbb{Q}, X > Y \iff \exists z \in \mathbb{Q}^+\) so that \(X = Y + z\).

6.6.7. Exercises: Referring to Definition 6.5.10,

1. \(X > 0 \iff X\) is positive.
2. \(X < 0 \iff X\) is negative.

6.6.8. Exercise: Referring to Definition 6.4.1, if \(x, y \in \mathbb{Q}^+\), then
\[
x < y \implies f(x) < f(y)
\]

6.7. Absolute Value.

6.7.1. Definition: The absolute value of \(X \in \mathbb{Q}\) is defined to be

1. \(|X| = X\) if \(X \geq 0\)
2. \(|X| = -X\) if \(X < 0\)
6.7.2. Exercises: Given \( X, Y, Z \in \mathbb{Q} \),

1. \(|X| = |-X|
2. \(|X| = Y \implies\) either \( X = Y \) or \( X = -Y \)
3. \(|X| > Y \implies\) either \( X > Y \) or \( X < -Y \)
4. \(|X| < Y \implies both \( X < Y \) and \( X > -Y \)
5. \(|XY| = |X||Y|
6. \(|X + Y| \leq |X| + |Y|
7. \(|X - Y| \geq ||X| - |Y||

6.7.3. Exercise: If \( X > 0 \) and \( Y > 0 \) then \( \exists n \in \mathbb{P} \) so that \( X < nY \)

6.7.4. Remark: This is called the **Achimedean Property** of positive rational numbers. It says that no matter how small the steps are \((Y)\), enough steps \((n)\) can go a long way (farther than \(X\)).

6.8. Division.

6.8.1. Exercise: \( X \neq 0 \implies \exists ! U \in \mathbb{Q} \) so that \( XY = 1 \)

6.8.2. Definition: If \( XY = 1 \), then \( U \) is called the **reciprocal** of \( X \).

6.8.3. Notation:

1. \( \frac{1}{X} \) denotes the reciprocal of \( X \)
2. \( \frac{Y}{X} \) denotes \( Y \cdot \frac{1}{X} \).

6.8.4. Exercises: For \( X, Y, Z, W \in \mathbb{Q} \) with \( Y \neq 0 \) and \( W \neq 0 \),

1. \( \frac{X}{Y} = \frac{Z}{W} \iff XY = YZ \)
2. \( \frac{X}{Y} + \frac{Z}{W} = \frac{XW + YZ}{YW} \)
3. \( \frac{XZ}{YW} = \frac{XZ}{YW} \)
4. \( \frac{XY}{YW} = \frac{X}{Y} \)
5. \( \frac{1}{Y} > \frac{1}{W} \iff Y < W \)