CONSTRUCTION OF $\mathbb{R}$

Part 1. Preliminaries

1. Symbolic Logic

1.1. Statements.

1.1.1. Read Handout 1.

1.1.2. Learn the following notation

   (1) $\lor$ “or”
   (2) $\land$ “and”
   (3) $\neg$ “not”
   (4) $\rightarrow$ “implies”
   (5) $\implies$ “implies”
   (6) $\leftrightarrow$ “if and only if”
   (7) $\iff$ “if and only if”
   (8) $\forall$ “for all” or “for every”
   (9) $\exists$ “there exists”

1.1.3. Exercises.

   (1) Write down the symbolic negation of each of the following
      (a) $p \lor q$
      (b) $p \land q$
      (c) $\neg p$
      (d) $p \implies q$
      (e) $\forall x, p$
      (f) $\exists x, p$
      (g) $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \implies |a_n - L| < \epsilon$

   (2) Handout 1, Exercise 3, parts (a) through (f).
   (3) Handout 1, Exercise 4, all parts.

1.2. Truth Tables.

1.2.1. Read Handout 2 as needed for the exercises.
1.2.2. Exercises.

(1) Write truth tables for each of the following
   (a) $p$
   (b) $\neg p$
   (c) $p \lor q$
   (d) $p \land q$
   (e) $p \implies q$
   (f) $\neg p \lor q$

(2) Handout 2, Problems 1–4.
(3) Write truth tables for each of the following
   (a) $\neg q \implies \neg p$
   (b) $\neg q \land p \implies r$, if $r$ is known to be false.

1.2.3. Definition. Two symbolic expressions involving $p$ and $q$ are said to be logically equivalent if they have identical truth value for all possible values of $p$ and $q$. That is, if their truth tables end up the same.

1.2.4. Exercise. In Exercises 1 and 3 above (not from the handout) there are four expressions that are all logically equivalent. List them. Memorize them.

1.3. Valid Arguments.

1.3.1. Read Handout 3 as needed for the exercises.

1.3.2. Definition. (Alternate version of Definition 1.5.1 from Handout 3.) An argument with premises $p_1, \ldots, p_n$ and conclusion $q$ is valid if and only if

\[(p_1 \land \cdots \land p_n) \implies q\]

is always true.

1.3.3. Exercises.

(1) Confirm the validity of the five rules of inference in Handout 3. ($\textit{Modus ponens, modus tollens,}$ etc.)
(2) Handout 3, Problem 4(a).
(3) Handout 3, Problem 5, parts (a), (g), (j), (l) and (m).

2. Set Theory

2.1. Operations.

2.1.1. $\textit{Set}$ will be an undefined term. It is enough to know that a set has $\textit{elements}$, although we do not define that word either.
2.1.2. **Examples.** Sets can be specified by listing their elements.

(1) \( A = \{x, y, z, w\} \)
(2) \( B = \{x, \{y\}, t\} \)

2.1.3. **Notation.**

(1) \( \in \) “is in” or “is an element of”
(2) \( \not\in \) “is not in” or “is not an element of”

2.1.4. **Definitions.** Most sets are defined by a rule that states a membership condition. Frequently this condition will refer to other sets. Some examples:

(1) The **intersection** of \( A \) and \( B \) is the set \( A \cap B \) defined by

\[
x \in A \cap B \iff x \in A \text{ and } x \in B
\]

(2) The **union** of \( A \) and \( B \) is the set \( A \cup B \) defined by

\[
x \in A \cup B \iff x \in A \text{ or } x \in B
\]

(3) The **set difference** of \( A \) and \( B \) is the set \( A \setminus B \) defined by

\[
x \in A \setminus B \iff x \in A \text{ and } x \not\in B
\]

(4) The **cross product** of \( A \) and \( B \) is the set \( A \times B \) defined by

\[
(x, y) \in A \times B \iff x \in A \text{ and } y \in B
\]

2.1.5. **Exercises.** With \( A \) and \( B \) as in the Example above, write out the elements of each of the following sets:

(1) \( A \cap B \)
(2) \( A \cup B \)
(3) \( A \setminus B \)
(4) \( A \times B \).

2.2. **Proofs.**

2.2.1. **Definition.** \( A \) is a **subset** of \( B \) if and only if \( \forall x \in A, x \in B \).

2.2.2. **Notation.** We write \( A \subset B \). This is commonly read “is a subset of” or “is contained in”
2.2.3. Exercises.

(1) Given: \( A \subset B \) and \( B \subset C \).
    Prove: \( A \subset C \).
(2) Given: \( A \subset B \) and \( B \subset C \).
    Prove: \( A \cup B \subset C \).
(3) Given: \( A \subset B \) and \( A \subset C \).
    Prove: \( A \subset B \cap C \).
(4) Prove that \( A \subset C \) and \( B \subset D \Rightarrow A \times B \subset C \times D \).
(5) Prove that \( A \times B \subset C \times D \Rightarrow A \subset C \) and \( B \subset D \).

2.2.4. Definition. Sets \( A \) and \( B \) are equal if and only if \( A \subset B \) and \( B \subset A \).

2.2.5. Notation. We say (obviously) \( A = B \).

2.2.6. Exercises.

(1) Prove that \( A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \).
(2) Prove that \( A \cup B \subset A \cap B \Rightarrow A = B \).
(3) Prove that \( A \times (B \cup C) = (A \times B) \cup (A \times C) \).
(4) Prove that \( A \times (B \cap C) = (A \times B) \cap (A \times C) \).
(5) Prove that \( A \times (B \setminus C) = (A \times B) \setminus (A \times C) \).
Part 2. Natural Numbers

3. Positive Integers

3.1. Peano’s Postulates. From now on the letter \( P \) will denote a set with the following properties:

1. \( 1 \in P \).
2. \( \forall n \in P, \exists \) unique \( n' \in P \) called the successor of \( n \).
3. \( \forall n \in P, n' \neq 1 \).
4. \( \forall n, m \in P, n' = m' \implies n = m \).
5. If \( Q \subset P \) satisfies
   (a) \( 1 \in Q \) and
   (b) \( n \in Q \implies n' \in Q \).
   then \( P = Q \).

3.1.1. Exercises.

1. Prove that \( \forall n, m \in P, n = m \implies n' = m' \).
2. Prove that \( \forall n \in P, n \neq n' \).
3. Prove that \( \forall n \neq 1 \in P \), there exists \( m \in P \) with \( m' = n \).

3.1.2. Definition: If \( n \in P \), the predecessor of \( n \) is any \( m \in P \) so that \( m' = n \).

3.1.3. Note: The preceding exercise proves that there is a predecessor to every element of \( P \) except 1.

3.1.4. Exercise: Prove that \( \forall n \neq 1 \in P \), the predecessor of \( n \) is unique.

3.2. Addition

3.2.1. Definition: \( \forall n, m \in P \),
   1. \( n + 1 = n' \)
   2. \( n + m' = (n + m)' \).

3.2.2. Exercise: Prove that \( \forall n, m \in P, \exists k \in P \) with \( k = n + m \)

3.2.3. Notation: \( \exists ! \) means “There exists a unique”. Note that this means the previous exercise is actually two problems.

3.2.4. Exercises: Prove for all \( n, m, k \in P \):
   1. \( (n + m) + k = n + (m + k) \)
   2. \( 1 + n = n + 1 \)
   3. \( n + m = m + n \)
   4. \( n \neq n + m \)
   5. \( n \neq m \implies n + k \neq m + k \)
   6. \( n = m \implies n + k = m + k \)
3.3. Ordering of $P$.

3.3.1. Exercises: Given $n, m \in P$, consider the following statements:

$p$: $n = m$
$q$: $\exists k \in P$ so that $n = m + k$
$r$: $\exists k \in P$ so that $m = n + k$

(1) Prove that $p \vee q \vee r$ is true.
(2) Prove that $p \land q$ is false. [Hint: Try proof by contradiction.]
(3) Prove that $q \land r$ is false. [Hint: Try proof by contradiction.]
(4) Prove that $p \land r$ is false. [Hint: Try proof by contradiction.]

3.3.2. Remark: This exercise is usually phrased as follows

Prove that exactly one of $p$, $q$ and $r$ is true.”

3.3.3. Definition: $\forall n, m \in P$,

(1) $n > m \iff \exists k \in P$ so that $n = m + k$.
(2) $n < m \iff \exists k \in P$ so that $m = n + k$.

3.3.4. Exercises: $\forall n, m \in P$, prove that

(1) Exactly one of $n = m$, $n > m$, and $n < m$ is true.
(2) $n > m \iff m < n$

3.3.5. Note: This is usually called the Trichotomy Law for $<$ on $P$, or just trichotomy when the context is clear.

3.3.6. Definition: $\forall n, m \in P$.

(1) $n \geq m \iff n > m$ or $n = m$
(2) $n \leq m \iff n < m$ or $n = m$

3.3.7. Exercises: $\forall n, m, k \in P$, prove that

(1) $n \geq m \iff m \leq n$
(2) $n < m$ and $m < k \implies n < k$
(3) $n > m$ and $m > k \implies n > k$
(4) $n \leq m$ and $m < k \implies n < k$
(5) $n \geq m$ and $m > k \implies n > k$
(6) $n < m$ and $m \leq k \implies n < k$
(7) $n > m$ and $m \geq k \implies n > k$
(8) $n \leq m$ and $m \leq k \implies n \leq k$
(9) $n \geq m$ and $m \geq k \implies n \geq k$
(10) $n + m > n$
(11) $n \geq m \implies n + k \geq m + k$
(12) $n \leq m \implies n + k \leq m + k$
(13) $n > m \implies n + k > m + k$
(14) $n < m \implies n + k < m + k$
(15) \(n + k \geq m + k \implies n \geq m\)
(16) \(n + k \leq m + k \implies n \leq m\)

3.3.8. Exercises: \(\forall n, m, l, k \in P\), prove that
(1) \(n > m\) and \(k > l\) implies \(n + k > m + l\).
(2) \(n \geq m\) and \(k > l\) implies \(n + k > m + l\).
(3) \(n > m\) and \(k \geq l\) implies \(n + k > m + l\).
(4) \(n \geq m\) and \(k \geq l\) implies \(n + k \geq m + l\).

3.3.9. Remark: The exercises in 3.3.7 are true with all the inequalities reversed as well. From now on, assume that that is true whenever it is needed.

3.3.10. Exercises: \(\forall n, m, k \in P\), prove that
(1) \(n \geq 1\)
(2) \(n > m \implies n \geq m'\).
(3) \(n < m' \implies n \leq m\).

4. Multiplication

4.1. Existence.

4.1.1. Definition: \(\forall n, m \in P\)
(1) \(n \cdot 1 = n\)
(2) \(n \cdot m' = (n \cdot m) + n\)

4.1.2. Exercise: \(\forall n, m \in P\), \(\exists ! k \in P\) with \(n \cdot m = k\).

4.1.3. Exercises: \(\forall n, m, k \in P\)
(1) \(1 \cdot n = n \cdot 1\)
(2) \(n \cdot (m + k) = n \cdot m + n \cdot k\)
(3) \((n + m) \cdot k = n \cdot k + m \cdot k\)

4.1.4. Exercises: \(\forall n, m, k \in P\)
(1) \(n \cdot m = m \cdot n\)
(2) \((n \cdot m) \cdot k = n \cdot (m \cdot k)\).

4.1.5. Exercises: \(\forall n, m, k \in P\)
(1) \(n > m \iff n \cdot k > m \cdot k\)
(2) \(n = m \iff n \cdot k = m \cdot k\)
(3) \(n < m \iff n \cdot k < m \cdot k\)
4.1.6. **Exercises:** \( \forall n, m, k, l \in P \)

(1) \( n > m \) and \( k > l \) \( \implies \) \( n \cdot k > m \cdot l \)

(2) \( n > m \) and \( k \geq l \) \( \implies \) \( n \cdot k \geq m \cdot l \)

(3) \( n \geq m \) and \( k \geq l \) \( \implies \) \( n \cdot k \geq m \cdot l \)

4.1.7. **Remark:** These results hold with all the inequalities reversed as well. Although we will not prove them, we will assume them whenever needed.

4.1.8. **Remark:** From this point forward, we are allowed all of the familiar algebraic manipulations on elements of \( P \), with the notable exception of **subtraction** and **division**.

4.1.9. **Notation:** From now on we will denote multiplication in \( P \) by the usual algebraic method of juxtaposition. I.e.,

\[ n \cdot m = nm \]
Part 3. Rational Numbers

5. Positive Rationals

5.1. Equivalence. Recall that \( P \times P \) is the set of all ordered pairs \((n, m)\) with \( n, m \in P \).

5.1.1. Definition. \( \forall (n, m), (k, l) \in P \times P \)

\[ (n, m) \sim (k, l) \iff nl = mk \]

5.1.2. Exercises: \( \forall (n, m), (k, l), (p, q) \in P \times P \)

1. \( (n, m) \sim (n, m) \)

2. \( (n, m) \sim (k, l) \implies (k, l) \sim (n, m) \)

3. \( (n, m) \sim (k, l) \) and \( (k, l) \sim (p, q) \implies (n, m) \sim (p, q) \)

5.1.3. Remark: The previous exercise is meant to show that “\( \sim \)” can be read as “is equivalent to”. More about this later.

5.1.4. Exercise: \( \forall n, m, k \in P, (n, m) \sim (nk, mk) \)

5.1.5. Definition: \( (n, m) > (k, l) \iff nl > mk \)

5.1.6. Exercise: Given \((n, m), (k, l) \in P \times P\), exactly one of the following is true:

1. \( (n, m) \sim (k, l) \)

2. \( (n, m) > (k, l) \)

3. \( (k, l) > (n, m) \)

5.2. Fractions.

5.2.1. Definition: \( \forall n, m \in P, \)

\[ \frac{n}{m} = \{(k, l) \in P \times P : (n, m) \sim (k, l)\} \]

5.2.2. Notation: We will also write \( \frac{n}{m} \) or \( n/m \) for \( \frac{n}{m} \).

Note that \( n/m \) is a set. Recall set equality proofs from Section 2.2 for the following

5.2.3. Exercises:

1. \( (n, m) \sim (k, l) \implies \frac{n}{m} = \frac{k}{l} \)

2. \( \forall (n, m), (k, l) \in P \times P \), exactly one of the following is true:

   a. \( \frac{n}{m} = \frac{k}{l} \)

   b. \( \frac{n}{m} \cap \frac{k}{l} = \emptyset \)
5.2.4. \textbf{Definition}: The \textbf{Positive Rational Numbers} are \[ \mathbb{Q}^+ = \left\{ \frac{n}{m} : n, m \in P \right\} \]

Note that any one positive rational number is a \textit{set}. However, all manipulations of a given rational number require a choice of some pair \((n, m) \in P \times P\) that \textit{represents} the rational number.

5.2.5. \textbf{Definition}: Given \(\frac{n}{m}, \frac{k}{l} \in \mathbb{Q}^+\)

\begin{align*}
(1) \quad \frac{n}{m} + \frac{k}{l} &= \frac{nl + mk}{ml} \\
(2) \quad \frac{n}{m} \cdot \frac{k}{l} &= \frac{nk}{ml} \\
(3) \quad \frac{n}{m} > \frac{k}{l} &\iff nl > mk
\end{align*}

This definition requires some additional explication. Consider the first item, which describes the addition of two \textit{sets}.

\begin{itemize}
  \item The definition states the the result of addition is yet another set.
  \item That set is determined by a \textit{representative} pair in \(P \times P\).
  \item However, that pair must be built from representative pairs for the sets being added.
  \item It would appear that the addition of two set depends not only on the sets, but on \textit{the particular choice of representatives} for each set.
\end{itemize}

Immediately after giving a definition that depends on an implicit choice, we must prove that the choice is not relevant. We will do this for all three definitions.

5.2.6. \textbf{Exercises}: Suppose that \(\frac{n}{m}, \frac{k}{l}, \frac{p}{q}\) are elements of \(\mathbb{Q}^+\).

\begin{align*}
(1) & \quad \text{Given } \frac{n}{m} = \frac{p}{q}, \text{ prove that } \frac{n}{m} + \frac{k}{l} = \frac{p}{q} + \frac{k}{l}. \\
(2) & \quad \text{Given } \frac{k}{l} = \frac{p}{q}, \text{ prove that } \frac{n}{m} + \frac{k}{l} = \frac{n}{m} + \frac{p}{q}. \\
(3) & \quad \text{Given } \frac{n}{m} = \frac{p}{q}, \text{ prove that } \frac{n}{m} \cdot \frac{k}{l} = \frac{p}{q} \cdot \frac{k}{l}. \\
(4) & \quad \text{Given } \frac{k}{l} = \frac{p}{q}, \text{ prove that } \frac{n}{m} \cdot \frac{k}{l} = \frac{n}{m} \cdot \frac{p}{q}. \\
(5) & \quad \text{Given } \frac{n}{m} = \frac{p}{q}, \text{ prove that } \frac{n}{m} > \frac{k}{l} \iff \frac{p}{q} > \frac{k}{l}. \\
(6) & \quad \text{Given } \frac{k}{l} = \frac{p}{q}, \text{ prove that } \frac{n}{m} > \frac{k}{l} \iff \frac{n}{m} > \frac{p}{q}.
\end{align*}
This proves that addition, multiplication and ordering are well defined on \( \mathbb{Q}^+ \). From now on we may speak of \( x \in \mathbb{Q}^+ \), with the freedom to choose a representative at any time.

5.2.7. **Definition**: \( \forall x, y \in \mathbb{Q}^+ \)

(1) \( x < y \iff y > x \)
(2) \( x \leq y \iff x < y \) or \( x = y \)
(3) \( x \geq y \iff y \leq x \)

5.2.8. **Exercise**: Given \( x, y \in \mathbb{Q}^+ \), prove that exactly one of the following is true:

(1) \( x > y \)
(2) \( x = y \)
(3) \( x < y \)

5.2.9. **Exercise**: Given \( x, y \in \mathbb{Q}^+ \)

(1) \( x > y \iff \exists z \in \mathbb{Q}^+ \) so that \( x = y + z \).
(2) \( x < y \iff \exists z \in \mathbb{Q}^+ \) so that \( y = x + z \).

5.3. **Algebra of Positive Rationals**.

5.3.1. **Exercises**: \( \forall x, y, z, w \in \mathbb{Q}^+ \),

(1) \( x + y = y + x \)
(2) \( x + (y + z) = (x + y) + z \)
(3) \( x \cdot y = y \cdot x \)
(4) \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \)
(5) \( x \cdot (y + z) = x \cdot y + x \cdot z \)
(6) \( (x + y) \cdot z = x \cdot z + y \cdot z \)
(7) \( x = y \iff x + z = y + z \)
(8) \( x > y \iff x + z > y + z \)
(9) \( x \geq y \) and \( z \geq w \implies x + z \geq y + w \)
(10) \( x = y \iff x \cdot z = y \cdot z \)
(11) \( x > y \iff x \cdot z > y \cdot z \)
(12) \( x \geq y \) and \( z \geq w \implies x \cdot z \geq y \cdot w \)

5.3.2. **Remark**: There are many possible permutations of this exercise. We will not prove them all, but simply assume from now on that all the familiar algebraic operations are legal for positive rationals.

5.4. **Inclusion of Natural Numbers**. This is a short section, devoted to unifying the two number systems we have built.

5.4.1. **Definition**: Let \( f : \mathbb{P} \to \mathbb{Q}^+ \) be the function given by

\[
f(n) = \frac{n}{1}
\]
5.4.2. Exercises: \( \forall n, m \in P \), prove that

1. \( f(n) = f(m) \Rightarrow n = m \)
2. \( f(n + m) = f(n) + f(m) \)
3. \( f(nm) = f(n) \cdot f(m) \)
4. \( n < m \iff f(n) < f(m) \)

5.4.3. Remarks: The formal name for this exercise is that \( P \) embeds in \( \mathbb{Q}^+ \). The function \( f \) is called the embedding. The set of rational numbers in the range of \( f \) is usually denoted \( f(P) \).

The first exercise says no two integers get mapped to the same rational, so \( f(P) \) is the same size as \( P \). The later exercises say that algebra in \( f(P) \) using rational “+”, “·”, and “<” is identical to algebra in \( P \) using the other “+”, “·”, and “<”.

Because of that we say that \( P \) is identified with \( f(P) \), and in practice adopt the following conventions:

1. For all \( n \in P \), we use \( n \) to denote \( n/1 \in \mathbb{Q}^+ \).
2. For all \( n \in P \) and \( x \in \mathbb{Q}^+ \):
   a. \( n + x \) means \( \frac{n}{1} + x \)
   b. \( nx \) means \( \frac{n}{1} \cdot x \)
   c. \( n > x \) means \( \frac{n}{1} > x \)
3. If \( x, y \in \mathbb{Q}^+ \), \( xy \) is the same as \( x \cdot y \)

Note also that there is a number called “1” in the set \( \mathbb{Q}^+ \) with some familiar properties:

5.4.4. Exercise:

1. \( \forall x \in \mathbb{Q}^+ \), \( x \cdot 1 = x \) and \( x = 1 \cdot x \)
2. \( \forall n \in P \), \( n \cdot \frac{1}{n} = 1 \)
3. \( \forall x \in \mathbb{Q}^+ \), \( \exists u \in \mathbb{Q}^+ \) so that \( xu = 1 \).

We now have a complete algebraic system that includes natural numbers and positive rational numbers with all the expected rules of algebra.
6. **Negative Rationals**

6.1. **Overview.** Our construction of $\mathbb{Q}^+$ could be summarized as follows:

1. Define an equivalence relation on $P \times P$.
2. Define $\mathbb{Q}^+$ to be the set of equivalence classes.
3. Prove that algebra exists and behaves normally.
4. Embed $P$ into $\mathbb{Q}^+$.

We will construct the full set of rational numbers from $\mathbb{Q}^+$ in the same way.

6.2. **Equivalence.**

6.2.1. **Definition:** $\forall (x, y), (z, w) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, 
\[(x, y) \sim (z, w) \iff x + w = y + z\]

6.2.2. **Exercises:** $\forall (x, y), (z, w), (u, v) \in \mathbb{Q}^+ \times \mathbb{Q}^+$

1. $(x, y) \sim (x, y)$
2. $(x, y) \sim (z, w) \implies (z, w) \sim (x, y)$
3. $(x, y) \sim (z, w)$ and $(z, w) \sim (u, v) \implies (x, y) \sim (u, v)$

6.2.3. **Definition:** $\forall x, y \in \mathbb{Q}^+$,  
\[ [x, y] = \{ (z, w) \in \mathbb{Q}^+ \times \mathbb{Q}^+ : (x, y) \sim (z, w) \} \]

6.2.4. **Exercises:** $\forall (x, y), (z, w) \in \mathbb{Q}^+ \times \mathbb{Q}^+$:

1. $(x, y) \sim (z, w) \iff [x, y] = [z, w]$
2. Exactly one of $[x, y] = [z, w]$ or $[x, y] \cap [z, w] = \emptyset$ is true.

6.2.5. **Definition:** The **Rational Numbers** are  
\[ \mathbb{Q} = \{ [x, y] : x, y \in \mathbb{Q}^+ \} \]

6.2.6. **Note:** Any rational number $[x, y]$ is a set. Any pair $(x, y)$ in the set is said to **represent** $[x, y]$.

6.2.7. **Definition:** Given $[x, y], [z, w] \in \mathbb{Q}$

1. $[x, y] + [z, w] = [x + z, y + w]$
2. $[x, y] \cdot [z, w] = [xz + yw, xw + yz]$

As before, these operations are performed on sets, but depend on representatives of each set. Thus we cannot continue until we prove that they are well defined. That is, that the answers are independent of the choice of representative.
6.2.8. Exercises: Suppose that \([x, y], [z, w], [u, v]\) are elements of \(\mathbb{Q}\).

(1) Given \([x, y] = [u, v]\), prove that \([x, y] + [z, w] = [u, v] + [z, w]\).
(2) Given \([z, w] = [u, v]\), prove that \([x, y] + [z, w] = [x, y] + [u, v]\).
(3) Given \([x, y] = [u, v]\), prove that \([x, y] \cdot [z, w] = [u, v] \cdot [z, w]\).
(4) Given \([z, w] = [u, v]\), prove that \([x, y] \cdot [z, w] = [x, y] \cdot [u, v]\).

From now on we may speak of \(X \in \mathbb{Q}\), with the freedom to choose a representative at any time.

6.3. Algebra.

6.3.1. Exercises: \(\forall X, Y, Z \in \mathbb{Q}\)

(1) \(X + Y = Y + X\)
(2) \(X + (Y + Z) = (X + Y) + Z\)
(3) \(X \cdot Y = Y \cdot X\)
(4) \(X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z\)
(5) \(X \cdot (Y + Z) = X \cdot Y + X \cdot Z\)
(6) \((X + Y) \cdot Z = X \cdot Z + Y \cdot Z\)
(7) \(X = Y \iff X + Z = Y + Z\)
(8) \(X = Y \implies XZ = YZ\)

6.3.2. Remark: Note the lack of “\(\iff\)” in Exercise (8) above. Is is a useful exercise to try to prove

\[XZ = YZ \implies X = Y\]

You can’t, but getting stuck trying is worth the trouble.

6.4. Embedding the Positive Rationals.

6.4.1. Remarks: From now on we can mix algebra from all three systems by embedding in the largest. For example, if \(n \in P\), \(x \in \mathbb{Q}^+\) and \(Y \in \mathbb{Q}\), the expression

\[x(n + Y)\]

makes sense. It is just

\[
\left[ x + \frac{1}{1} \cdot \frac{1}{1} \right] \cdot \left( \left[ \frac{n + 1}{1} \right] + Y \right)
\]

We can also stop using \(\cdot\) for multiplication in \(\mathbb{Q}\).

6.5. Subtraction. In this section we will construct zero, and prove that subtraction exists.

6.5.1. Definition: Given \(x, y \in \mathbb{Q}^+\) and \(Z \in \mathbb{Q}\),

(1) \(-[x, y] = [y, x]\)
(2) \(0 = [x, x]\).
6.5.2. **Remark:** As usual, these definitions seem to depend on which specific $x$ or $(x, y)$ represents the set in question.

6.5.3. **Exercise:** Prove that the definitions do not depend on the choice of representative. Part of this exercise is leering how to state and prove “well defined”. Use Exercises 5.2.6 ?? as a guide.

6.5.4. **Remarks:**

1. We now have a thing called $0 \in \mathbb{Q}$.
2. For any $X \in \mathbb{Q}$, there is a rational number called $-X$.
3. Given $n \in P$ or $x \in \mathbb{Q}^+$, we can also make sense of $-n$ and $-x$.

6.5.5. **Definition:** Given $X, Y \in \mathbb{Q}$

$$X - Y = X + (-Y)$$

6.5.6. **Exercises:** Given $X, Y, Z \in \mathbb{Q}$

1. $X + 0 = X$
2. $X - X = 0$
3. $X(Y - Z) = XY - XZ$
4. $(X - Y)Z = XZ - YZ$
5. $-(-X) = X$
6. $(-1)(X) = -X$
7. $(-X)(-Y) = XY$

6.5.7. **Exercises:** Given $X, Y \in \mathbb{Q}$

1. $X = -X \iff X = 0$
2. $XY = 0 \iff X = 0$ or $Y = 0$

6.5.8. **Remark:** If $x \in \mathbb{Q}^+$, $x$ also denotes the rational number $[x+1, 1]$. It follows that $-x$ denotes the rational number $[1, x + 1]$.

6.5.9. **Exercise:** Suppose $X \in \mathbb{Q}$. If $X \neq 0$ then there exists $z \in \mathbb{Q}^+$ so that exactly one of the following is true.

1. $X = z$
2. $X = -z$

6.5.10. **Definition:** If the first condition occurs, we say $X$ is **positive**. If the second condition occurs, we say $X$ is **negative**.
6.6. Ordering.

6.6.1. Definition: Given \([x, y]\) and \([z, w] \in \mathbb{Q}^+\),
\[ [x, y] < [z, w] \iff x + w < z + y \]

6.6.2. Exercise: Prove that this is well defined.

6.6.3. Definition: \(\forall X, Y \in \mathbb{Q}\)
(1) \(X > Y \iff Y < X\)
(2) \(X \leq Y \iff X = Y \text{ or } X < Y\)
(3) \(X \geq Y \iff Y \leq X\)

6.6.4. Exercise: For all \(X, Y \in \mathbb{Q}\), prove that exactly one of \(X = Y\), \(X < Y\), \(X > Y\) is true.

6.6.5. Exercises: For all \(X, Y, Z, W \in \mathbb{Q}\)
(1) \(X \geq Y \text{ and } Y \geq Z \implies X \geq Z\).
(2) \(X > Y \iff X + Z > Y + Z\)
(3) \(X \geq Y \text{ and } Z \geq W \implies X + Z \geq Y + W\)
(4) If \(Z > 0\), then \(X > Y \iff XZ > YZ\)
(5) If \(Z < 0\), then \(X > Y \iff XZ < YZ\)
(6) If \(Z \neq 0\), then \(X = Y \iff XZ = YZ\)

6.6.6. Exercise: \(\forall X, Y \in \mathbb{Q}, X > Y \iff \exists z \in \mathbb{Q}^+\) so that \(X = Y + z\).

6.6.7. Exercises: Referring to Definition 6.5.10,
(1) \(X > 0 \iff X\) is positive.
(2) \(X < 0 \iff X\) is negative.

6.6.8. Exercise: Referring to Definition 6.4.1, if \(x, y, \in \mathbb{Q}^+\), then
\[ x < y \implies f(x) < f(y) \]

6.7. Absolute Value.

6.7.1. Definition: The **absolute value** of \(X \in \mathbb{Q}\) is defined to be
(1) \(|X| = X\) if \(X \geq 0\)
(2) \(|X| = -X\) if \(X < 0\)
6.7.2. Exercises: Given $X, Y, Z \in \mathbb{Q}$,

1. $|X| = |-X|
2. $|X| = Y \implies$ either $X = Y$ or $X = -Y$
3. $|X| > Y \implies$ either $X > Y$ or $X < -Y$
4. $|X| < Y \implies$ both $X < Y$ and $X > -Y$
5. $|XY| = |X||Y|
6. $|X + Y| \leq |X| + |Y|
7. $|X - Y| \geq ||X| - |Y||$

6.7.3. Exercise: If $X > 0$ and $Y > 0$ then $\exists n \in P$ so that $X < nY$

6.7.4. Remark: This is called the **Achimedean Property** of positive rational numbers. It says that no matter how small the steps are ($Y$), enough steps ($n$) can go a long way (farther than $X$).

6.8. Division.

6.8.1. Exercise: $X \neq 0 \implies \exists ! U \in \mathbb{Q}$ so that $XU = 1$

6.8.2. Definition: If $XU = 1$, then $U$ is called the **reciprocal** of $X$.

6.8.3. Notation:

1. $\frac{1}{X}$ denotes the reciprocal of $X$
2. $\frac{Y}{X}$ denotes $Y \cdot \frac{1}{X}$.

6.8.4. Exercises: For $X, Y, Z, W \in \mathbb{Q}$ with $X \neq 0$ and $W \neq 0$,

1. $\frac{X}{Y} = \frac{Z}{W} \iff XW = YZ$
2. $\frac{X}{Y} + \frac{Z}{W} = \frac{XW + YZ}{YW}$
3. $\frac{XZ}{YW} = \frac{XZ}{YW}$
4. $\frac{XW}{YW} = \frac{X}{Y}$
5. If $YW > 0$, then $\frac{1}{Y} > \frac{1}{W} \iff Y < W$
Part 4. Real Numbers

7. Sequences

7.1. Introduction to Sequences.

7.1.1. Definition: A sequence of rational numbers is a function \( f \) from \( P \) to \( \mathbb{Q} \).

7.1.2. Examples:

(1) \( f(n) = n \)
(2) \( g(n) = \frac{1}{n} \)
(3) \( h(n) = \frac{n}{n + 1} \)

7.1.3. Notation:

(1) \( f_n \) is another notation for \( f(n) \).
(2) The entire sequence is referred to as \( f \) or sometimes \( \{f_n\} \)
(3) Sometimes sequences are written out term by term, for example

\[
h = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots \right\}
\]

7.2. Limits.

7.2.1. Remark: A limit is not something that you compute. Instead, a limit is something that you propose, and then verify by proof.

7.2.2. Definition: Suppose that \( f \) is a sequence and \( L \) is a rational number. The \textbf{limit of} \( f \text{ is } L \) if and only if

\[
\forall \epsilon \in \mathbb{Q}^+, \exists N \in P \text{ so that } n > N \implies |f_n - L| < \epsilon
\]

7.2.3. Notation: If the limit of \( f \) is \( L \) we write \( \lim f = L \) or \( f \to L \).

We will also say that \( f \text{ converges to } L \).

7.2.4. Remark: Suppose you have a sequence \( f \) with a proposed limit of \( L \). A proof of this must be structured as follows:

Begin with the sentence, “Choose \( \epsilon > 0. \)”
Construct an appropriate \( N \). (Techniques will vary.)
Write the sentence, “Given \( n > N. \)”
Write a proof that ends with

\[
|f_n - L| < \epsilon
\]
7.2.5. Exercises: Prove the following:
(1) $\frac{1}{n} \to 0$
(2) $\frac{n}{n + 1} \to 1$
(3) $\frac{n}{(n + 1)(n + 2)} \to 0$
(4) $\frac{n(n + 1)}{(2n + 1)(n + 2)} \to \frac{1}{2}$
(5) $\frac{1}{n^2} \to 0$
(6) $\frac{n}{n + 1} + \frac{1}{n^2} \to 1$

7.2.6. Exercise: Write down the negation of the statement
\[ \forall \epsilon \in \mathbb{Q}^+, \exists N \in P \text{ so that } n > N \implies |f_n - L| < \epsilon \]

7.2.7. Exercise: Suppose $f(n) = n$. Prove that $\forall L \in \mathbb{Q}$, $f$ does not converge to $L$.

7.3. Other Sequence Properties.

7.3.1. Definition: A sequence is bounded if there exists $B \in \mathbb{Q}$ so that for all $n$, $|f(n)| < B$.

7.3.2. Exercise: If $f \to L$ then $f$ is bounded.

7.3.3. Definition: Suppose that $g$ is a sequence in which $g_n$ is never zero. Then $1/g$ denotes a sequence defined by
\[ (1/g)(n) = \frac{1}{g(n)} \]

7.3.4. Exercise: If $g(n)$ is never zero, and $g \to L$ with $L \neq 0$, then $1/g$ is bounded.

7.3.5. Definition: Suppose that $f$ and $g$ are sequences and that $c$ is a rational number. Then $f + g$, $fg$, $cf$ and $|f|$ denote sequences defined by
(1) $(f + g)(n) = f(n) + g(n)$
(2) $(fg)(n) = f(n)g(n)$
(3) $(cf)(n) = cf(n)$
(4) $|f|(n) = |f(n)|$
7.3.6. *Exercises:* Suppose that $f \to L$ and $g \to M$. If $c$ is any rational number, prove that

1. $f + g \to L + M$
2. $cf \to cL$
3. $|f| \to |L|$
4. If $g(n)$ is never zero and $M \neq 0$, then $1/g \to 1/M$.
5. $fg \to LM$

7.4. **Cauchy Sequences.**

7.4.1. *Definition:* A sequence is a **Cauchy sequence** if

$$\forall \epsilon \in \mathbb{Q}^+, \exists N \in P \text{ so that } n > N \text{ and } m > N \implies |f_n - f_m| < \epsilon$$

7.4.2. *Exercises:*

1. Every Cauchy sequence is bounded.
2. Every convergent sequence is Cauchy.

7.4.3. *Exercises:* Suppose that $f$ and $g$ are Cauchy sequences and that $c$ is a rational number. Prove that:

1. $f + g$ is a Cauchy sequence.
2. $cg$ is a Cauchy sequence.
3. $|f|$ is a Cauchy sequence.
4. $fg$ is a Cauchy sequence.
8. Real Numbers

8.1. Equal Sequences. We usually know two things are equal simply because the “look the same”. Since an individual sequence is an infinite list of rationals, this is no longer entirely appropriate. Instead, we have

8.1.1. Definition. If $f$ and $g$ are sequences of rationals then $f = g$ if an only if

$$\forall n, f(n) = g(n)$$

8.1.2. Exercises: Suppose that $f, g, h$ are sequences of rational numbers. Prove that

1. $f + g = g + f$
2. $f + (g + h) = (f + g) + h$
3. $fg = gf$
4. $f(gh) = (fg)h$
5. $f(g + h) = fg + fh$
6. $(f + g)h = fh + gh$

8.2. Equivalent Sequences.

8.2.1. Definition. Suppose that $f$ and $g$ are Cauchy sequences of rational numbers.

$$f \sim g \iff \forall \varepsilon \in \mathbb{Q}^+, \exists N \in P \text{ so that } n > N \implies |f_n - g_n| < \varepsilon$$

8.2.2. Exercises: If $f, g$ and $h$ are Cauchy sequences of rational numbers, then

1. $f \sim f$
2. $f \sim g \implies g \sim f$
3. $f \sim g$ and $g \sim h \implies f \sim h$.

8.2.3. Definition: If $f$ is a Cauchy sequence of rationals, then

$$[f] = \{g : g \sim f\}$$

8.2.4. Exercises: If $f, g$ are Cauchy sequences of rationals, then exactly one of $[f] = [g]$ or $[f] \cap [g] = \emptyset$ is true.

8.2.5. Definition: The Real Numbers are

$$\mathbb{R} = \{[f] : f \text{ is a Cauchy sequence of rationals}\}$$

8.3. Algebra of Reals.

8.3.1. Definition: If $[f], [g] \in \mathbb{R}$

1. $[f] + [g] = [f + g]$
2. $[f] \cdot [g] = [fg]$
8.3.2. Exercise: Prove that these are well defined operations.

8.3.3. Exercises: \( \forall [f], [g], [h] \in \mathbb{R}, \)

1. \( [f] + [g] = [g] + [f] \)
2. \( [f] + ([g] + [h]) = ([f] + [g]) + [h] \)
3. \( [f] \cdot [g] = [g] \cdot [f] \)
4. \( [f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h] \)
5. \( [f] \cdot ([g] + [h]) = [f] \cdot [g] + [f] \cdot [h] \)
6. \( ([f] + [g]) \cdot [h] = [f] \cdot [h] + [g] \cdot [h] \)
7. \( [f] = [g] \implies [f] + [h] = [g] + [h] \)
8. \( [f] = [g] \implies [f] \cdot [h] = [g] \cdot [h] \)

8.3.4. Remark: Note the lack of cancellation exercises. This is because it makes more sense to develop the ideas of subtraction and division instead.

8.4. Embedding of \( \mathbb{Q} \).

8.4.1. Definition:

1. Given \( x \in \mathbb{Q} \), define the sequence \( f_x \) to be

\[ f_x(n) = x \]

2. Define \( \Phi : \mathbb{Q} \to \mathbb{R} \) as the function

\[ \Phi(x) = [f_x] \]

8.4.2. Exercises: Given \( x, y \in \mathbb{Q} \), prove that

1. \( f_x + f_y = f_{x+y} \)
2. \( f_x f_y = f_{xy} \)

8.4.3. Exercises: \( \forall x, y \in \mathbb{Q} \), prove that

1. \( \Phi(x) = \Phi(y) \implies x = y \)
2. \( \Phi(x + y) = \Phi(x) + \Phi(y) \)
3. \( \Phi(xy) = \Phi(x) \cdot \Phi(y) \)

8.4.4. Remark: As usual, this means that you can do algebra with any mixture of reals, rationals, etc. Sometimes you might have to interpret a rational number \( x \) as the sequence \( f_x \). It also means we can stop using "\( \cdot \)" for multiplication of reals.

8.5. Subtraction and Division.

8.5.1. Remark: There are no such things. Instead, there are additive inverses and multiplicative inverses.
8.5.2. **Exercises:** Suppose \([f] \in \mathbb{R}\).

1. Prove that \(\exists [g] \in \mathbb{R}\) so that \([f] + [g] = 0\).
2. Prove that this \([g]\) is unique. I.e., that \([f] + [h] = 0 \implies [h] = [g]\).

8.5.3. **Definition:** We call this number \(-[f]\), the **additive inverse** of \([f]\).

8.5.4. **Exercise:** Prove that \([f] + [h] = [g] + [h] \implies [f] = [g]\)

8.5.5. **Remark:** Multiplicative inverses are harder.

8.5.6. **Exercises:** Suppose that \([f] \in \mathbb{R}\) and \([f] \neq 0\).

1. Prove that \(\exists d \in \mathbb{Q}^+\) and \(\exists N \in \mathbb{P}\) so that \(n \geq N \implies |f_n| > d\).
2. With \(N\) as above, let \(g\) be any sequence of rationals such that \(g(n) = 1/f(n)\) for all \(n \geq N\). Prove that \(g\) is Cauchy.
3. With \(g\) as above, prove that \([f][g] = 1\).
4. If \([h] \in \mathbb{R}\) and \([f][h] = 1\), prove that \([h] = [g]\).

8.5.7. **Remark:** This proves that every non-zero \([f] \in \mathbb{R}\) has a unique **multiplicative inverse**, which we denote by \(1/[f]\).

8.5.8. **Exercise:** If \([h] \neq 0\), prove that \([f][h] = [g][h] \implies [f] = [g]\)
8.6. Ordering of $\mathbb{R}$.

8.6.1. *Definition:* For $[f], [g] \in \mathbb{R}$, define $[f] > [g]$ if and only if

$$\exists d \in \mathbb{Q}^+ \text{ and } \exists N \in P \text{ so that } n > N \implies f_n - g_n > d$$

8.6.2. *Exercises:*

1. Prove that $>$ is well defined.
2. Write down the negation of the definition of $>$.

8.6.3. *Exercises:* Let $f(n) = \frac{n}{n + 1}$ and $g(n) = \frac{1}{n}$. Prove that:

1. $[f] > [g]$
2. $[f] > 0$
3. $[g] \neq 0$
4. $h \rightarrow L \implies [h] \neq L$

8.6.4. *Exercise:* Prove that the embedding $\Phi : \mathbb{Q} \rightarrow \mathbb{R}$ preserves $<$.

8.6.5. *Exercise:* Prove that exactly one of $[f] > [g], [f] = [g], \text{ or } [g] > [f]$ is true.

8.6.6. *Exercises:* Given $[f], [g], [h], [k] \in \mathbb{R}$, prove that:

1. $[f] > [g] \text{ and } [g] > [h] \implies [f] > [h]$
2. $[f] > [g] \iff [f] + [h] > [g] + [h]$
3. $[f] > [g] \text{ and } [h] > [k] \implies [f] + [h] > [g] + [k]$
4. If $[h] > 0$, then $[f] > [g] \iff [f][h] > [g][h]$
5. If $[h] < 0$, then $[f] > [g] \iff [f][h] < [g][h]$
6. If $[f] > 0$ and $[g] > 0$, then $[f] > [g] \iff 1/[f] > 1/[g]$
7. If $[f] > 0$ and $[g] > 0$, then $\exists n \in P \text{ so that } n[g] > [f]$

8.7. Absolute Value.

8.7.1. *Definition:* For $[f] \in \mathbb{R}$, define

1. $|\lfloor f \rfloor| = [f] \text{ if } [f] \geq 0$
2. $|\lfloor f \rfloor| = -[f] \text{ if } [f] < 0$

8.7.2. *Remark:* All properties of absolute value for rational numbers (See 6.7.2) are true for real numbers. The proofs are identical, so we will skip them and assume the properties from now on.
9. Topology of \( \mathbb{R} \)

9.1 Sequences in \( \mathbb{R} \).

9.1.1. Definition: A sequence of real numbers is a function from \( P \) into \( \mathbb{R} \).

9.1.2. Notation: Notation is the same as for sequences of rationals, but we will usually use Greek letters to help distinguish the two contexts.

9.1.3. Examples: None, since we have no examples of real numbers that are not also rational numbers.

9.1.4. Definition: Suppose that \( \sigma \) is a sequence of reals and \( z \in \mathbb{R} \). The limit of \( \sigma \) is \( z \) if and only if

\[
\forall \epsilon > 0, \exists N \in P \text{ so that } n > N \implies |\sigma_n - z| < \epsilon
\]

9.1.5. Notes:

1. This is identical to the definition of limit for rationals, except that here \( \epsilon \) could be any positive real number.
2. All notation is also the same. For example, we write \( \sigma \to z \), and we say \( \sigma \) converges to \( z \).
3. As usual, whenever reals and rationals show up in the same context, all rationals are assumed to be transformed into reals the natural embedding.
4. Cauchy is also defined as for rationals, but with possibly real \( \epsilon \).

9.1.6. Definition: A sequence of reals, \( \sigma \), is Cauchy if and only if

\[
\forall \epsilon > 0, \exists N \in P \text{ so that } n, m > N \implies |\sigma_n - \sigma_m| < \epsilon
\]

9.1.7. Exercises:

1. Suppose that \( f \) is a sequence of rationals. Prove that \( f \to [f] \).
2. Suppose that \([f]\) and \([g]\) are real numbers with \([f] < [g]\). Prove that there exists \( x \in \mathbb{Q} \) with \([f] < x < [g]\).
3. Suppose that \( \sigma \) is a Cauchy sequence of reals. Prove that there exists a sequence of rationals \( f \) so that \( \sigma \to [f] \).

9.1.8. Remark: This last exercise is a deep result in the topology of \( \mathbb{R} \). It says that every Cauchy sequence of real numbers converges to a real number. By comparison, although we have not proved this, almost all Cauchy sequences in \( \mathbb{Q} \) converge to numbers outside of \( \mathbb{Q} \).
9.2. **The Monotonic Sequence Property.** The topology of the real numbers is characterized by the fact that sequences cannot "escape" from a finite region. It turns out that this is a consequence of Cauchy completeness and the Archimedean property.

9.2.1. **Definition:** Let \( \sigma \) be a sequence.

1. If \( \sigma_{n+1} \geq \sigma_n \) for all \( n \), then \( \sigma \) is **increasing**.
2. If \( \sigma_{n+1} \leq \sigma_n \) for all \( n \), then \( \sigma \) is **increasing**.
3. If \( \sigma \) is either increasing or decreasing, we say \( \sigma_n \) is **monotonic**.

9.2.2. **Exercises:** Suppose that \( \sigma \) is a sequence that is increasing and **not** Cauchy. That is,

\[ \exists \epsilon > 0 \text{ so that } \forall N, \exists n, m > N \text{ with } |\sigma_n - \sigma_m| \geq \epsilon \]

1. \( \forall N, \exists n > N \text{ so that } \sigma_n \geq \sigma_1 + N\epsilon. \)
2. Prove that \( \sigma \) is not bounded.

9.2.3. **Exercise:** (The Monotonic Sequence Property) Prove that every bounded, monotonic sequence converges.

9.3. **A Curious Example.** Define a sequence of rationals as follows:

1. \( f_1 = 2 \)
2. If \( f_n \neq 0 \), then \( f_{n+1} = \frac{1}{f_n} + \frac{f_n}{2} \).

9.3.1. **Exercise:** Prove that \( f_n > 0 \) for all \( n \).

9.3.2. **Remark:** This proves that \( f_n \) is defined for all \( n \).

9.3.3. **Exercises:** For all \( n \), prove:

1. \( f_{n+1} - f_n = \frac{1}{f_n} - \frac{f_n}{2} \).
2. \( f_{n+1}^2 - 2 = (f_{n+1} - f_n)^2 \).

9.3.4. **Exercises:** Prove that for all \( n \),

1. \( 1 \leq f_n \leq 2 \)
2. \( 2 < f_n^2 \)

9.3.5. **Exercises:**

1. Prove that \( f \) is monotonic.
2. Prove that \( f \) is bounded.

9.3.6. **Remark:** This proves that \( [f] \) is a real number.
9.3.7. Exercises: Define a new sequence $g$ by $g_1 = 0$ and $g_{n+1} = f_n$. Prove that:

1. $|g| = |f|$.
2. $f^2 = (g - f)^2 - 2$ (as sequences).
3. $|f|^2 = 2$.

9.3.8. Remark: This proves that there exists a real number called $\sqrt{2}$.

9.4. Irrational Numbers. Every rational number is also a real number, under the standard embedding $x \mapsto [f_x]$. In this section we will prove that not every real number is rational. First, some facts about even and odd integers.

9.4.1. Definition: Given $n \in P$,

1. $n$ is **even** if $\exists k \in P$ so that $n = 2k$.
2. $n$ is **odd** if $n = 1$ or if $\exists k \in P$ so that $n = 2k + 1$.

9.4.2. Note: If you need a “definition” of 2, use $2 = 1' = 1 + 1$.

9.4.3. Exercises:

1. Every $n \in P$ is either even or odd.
2. No $n \in P$ is both even and odd.
3. If $n$ is odd, then $n^2$ is odd.
4. If $n^2$ is even, then $n$ is even.

9.4.4. Exercises: Given $x \in \mathbb{Q}^+$,

1. There exist $n, m \in P$, not both even, so that $x = n/m$.
2. $x^2 \neq 2$.

This last exercise says that there is an algebraic limitation on the rational number system. Namely, that you cannot solve the equation $x^2 = 2$. However, in the previous section we proved that there is a solution in $\mathbb{R}$. This proves that the set $\mathbb{R} \setminus \Phi(\mathbb{Q})$ is not empty.

9.4.5. Definition: If $x$ is a real number and $x$ is not rational, we say $x$ is **irrational**.

9.4.6. Exercises: Let $x$ and $y$ be real numbers.

1. If $x$ is rational and $y$ is irrational, then $x + y$ and $xy$ are irrational.
2. If $x < y$, then there exists an irrational $z$ such that $x < y < z$. 