Homework 13

Suppose that $\gamma : [a, b] \to \mathbb{C}$ and $f : G \to \mathbb{C}$ are such that $\gamma'$ and $f \circ \gamma$ are continuous on $[a, b]$. We will prove the existence of the Riemann integral

$$\int_{\gamma} f(z) \, dz$$

There is some notation that we will use throughout the proof. If $P$ is a partition,

- $R_P$ will denote any one of the possible Riemann sums

$$R_P = \sum f(\gamma(z_i^*)) \Delta z_i = \sum f(\gamma(t_i^*)) [\gamma(t_i) - \gamma(t_{i-1})]$$

where each $t_i^*$ is arbitrarily chosen in $[t_{i-1}, t_i]$.

- $P'$ will denote a partition obtained from $P$ by inserting one more point. That is, $P' = P \cup \{t'\}$ where $t'$ is distinct from all other $t_n$ in $P$.

**Lemma.** (Proved in class) $\forall \epsilon > 0, \exists \delta > 0$ so that $||P|| < \delta \implies |R_{P'} - R_P| < \epsilon$.

**Exercise 1.** Prove that $\forall \epsilon > 0, \exists \delta > 0$ so that $||P|| < \delta \implies |R_Q - R_P| < \epsilon$ for all $Q$ with $P \subset Q$.

**Exercise 2.** Prove that $\forall \epsilon > 0, \exists \delta > 0$ so that $||P|| < \delta$ and $||Q|| < \delta$ $\implies |R_Q - R_P| < \epsilon$.

**Exercise 3.** Prove that there exists a sequence $\{\delta_n\}$ so that for all $n$

1. $\delta_{n+1} < \delta_n$, and
2. $||P||, ||Q|| < \delta_n \implies |R_P - R_Q| < 1/n$

**Exercise 4.** For each $\delta_n$ from Exercise 3, choose some $P_n$ with $||P_n|| < \delta_n$. Prove that $\{R_{P_n}\}$ is a Cauchy sequence, and that it therefore has a limit in $\mathbb{C}$.

**Exercise 5.** Suppose $R_{P_n} \to I$. Prove that $I$ is the Riemann integral

$$\int_{\gamma} f(z) \, dz$$

**Exercise 6.** Astute readers will note that $\{R_{P_n}\}$ is not uniquely defined. No problem. Suppose that some other number $J$ is also the Riemann integral. Prove that $I = J$. 1