Homework 10

1. Suppose that

\[ f(z) = \sum (-1)^n \frac{z^{2n+1}}{(2n + 1)!} \]

(a) Prove that \( f(z) \) is entire.
(b) Write \( f'(z) \) as a power series.
(c) Prove that \( f \) solves the differential equation \( y'' + y = 0 \).
(d) Prove that \( f \) solves the initial value problem \( y'' + y = 0, y(0) = 0, y'(0) = 1 \).
(e) The Fundamental Theorem of Differential Equations says that (at least in this case) there is a unique real valued solution to this IVP, and that that solution is defined on all of \( \mathbb{R} \). Find that solution (in non-series form).

NOTE: This means that, for real \( z \), \( f(z) \) is a familiar function.

2. Suppose that \( k \) is a fixed real number and that

\[ g(z) = \sum \frac{(kz)^n}{n!} \]

(a) Prove that \( g(z) \) is entire.
(b) Write \( g'(z) \) as a power series.
(c) Find a differential equation for which \( g \) would be a solution.
(d) Write down an initial value problem for which \( g \) would be a solution.
(e) Use the FTDE to determine \( g(z) \) (in non-series form) for real valued \( z \).

3. Fix \( k = 1 \) and let \( g(z) \) be defined as in Problem 2. Let

\[ h(z) = \sum (-1)^n \frac{z^{2n}}{(2n)!} \]

as in class. Prove that for all \( z, w \in bC \):

(a) \( g(z + w) = g(z)g(w) \).
(b) \( g(iz) = h(z) + if(z) \)
(c) \( |f(z)|^2 + |h(z)|^2 = 1 \)

Then write these conclusions using the familiar names for these functions.
4. Suppose that

\[ l(z) = \sum (-1)^n(z - 1)^n \]

(a) Note that this is a geometric series and compute:

i. The radius of convergence.
ii. An exact formula for \( l(z) \).

(b) Find a power series for a function \( L(z) \) so that

i. \( L'(z) = l(z) \), and
ii. \( L(1) = 0 \).

(c) Apply FTDE once again (the IVP is \( y'(t) = 1/t, y(1) = 0 \)) to guess a familiar name for \( L(z) \) when \( z \) is positive and real.

5. Let \( L \) be defined as in Problem 4. Suppose that there is a continuous \( g \) such that \( (L \circ g)(z) = z \) for all \( z \) in an open set \( G \). Prove that \( g' = g \) on the domain \( G \).

6. Fix \( k = 1 \) and let \( g(z) \) be defined as in Problem 2. Suppose that there is a continuous \( f \) for which \( |z - 1| < 1 \implies (g \circ f)(z) = z \). Prove that \( f = L \) from Problem 4.