1. (10 pts.) Find the number of integers between 1 and 500 (including 1 and 500) that are divisible by 9 but not divisible by 12.

Solution:

Let $A$ be the set of integers divisible by 9.
Let $B$ be the set of integers divisible by 12.
$A \cap B$ is the set of integers divisible by 36.
We need to count $A \cap B^c$.

$$|A \cup B^c| = |A| - |A \cap B|$$
$$= \left\lfloor \frac{500}{9} \right\rfloor - \left\lfloor \frac{500}{36} \right\rfloor$$
$$= 55 - 13$$
$$= 42$$

2. (10 pts.) How many permutations of the alphabet start with a vowel and end with a consonant? (Just to be clear, the vowels are $a, e, i, o,$ and $u$.)

Solution:

5 choices for the first letter.
21 choices for the last letter.
24! for the letters in between.

Answer: $5 \cdot 24! \cdot 21 = 105 \cdot 24!$
3. (10 pts.) How many three letter words contain both the letters “A” and “B”? (Assume that “word” means any three capital letters.)

Solution:

Easier to count the complement: words without “A” or without “B”.
Let \( X \) be the set of words without “A”.
Let \( Y \) be the set of words without “B”.
Note that \( X \cap Y \) is the set of words with neither of “A” or “B”.
Let \( Z \) be the set of all words.
We need to count \( |Z| - |X \cup Y| \).

\[
|Z| - |X \cup Y| = |Z| - |X| - |Y| + |X \cap Y|
= 26^3 - 25^3 - 25^3 + 24^3
= 150.
\]

4. (10 pts.) From a pile of 10 identical red balls and a pile of 20 identical green balls, how many ways can you choose a set of 5 balls so that exactly two are green.

Solution:

From 10, choose 3; from 20 choose 2.

\[
\binom{10}{3} \binom{20}{2} = \frac{10!}{7!3!} \cdot \frac{20!}{18!2!}
= \frac{10 \cdot 9 \cdot 8 \cdot 20 \cdot 19}{3 \cdot 2 \cdot 2}
= (10 \cdot 3 \cdot 4)(10 \cdot 19)
= 22,800.
\]
5. Consider the following algorithm:

Input: $n$
Set $P = 0$
For $i$ from 1 to $n$ {
  $P \rightarrow P + n$
}
Output: $P$

(a) (10 pts.) Execute the algorithm for $n = 5$.

Solution:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>

Output: 25

(b) (10 pts.) Compute a complexity function for the algorithm.

Solution:

1 operation for $P = 0$.
1 operation for each loop.
$n$ loops.
Total: $n + 1$. 
6. (10 pts.) Given a set of numbers \( a_1, a_2, a_3, \ldots a_n \), describe an algorithm to compute the geometric mean
\[
\sqrt[\text{n}]{a_1 a_2 \cdots a_n}
\]
Addition, subtraction, multiplication or division of two numbers is a “simple operation”, as is taking any square root.

Solution:

\textbf{Input:} \( a_1, a_2, a_3, \ldots a_n \)

Set \( P = 1 \)

For \( i \) from 1 to \( n \) {
\[
P \rightarrow P \times a_i
\]
}

\( A \rightarrow \sqrt{P} \)

7. (10 pts.) Sketch a graph that has degree sequence 3,2,2,1.

Solution:

[Graph image]

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8. (20 pts.) Two of the following graphs are isomorphic. Determine which two. Justify the isomorphism and explain why the third one is not isomorphic.

A. [Graph A]
B. [Graph B]
C. [Graph C]

Solution:

A is bipartite.
So is C, if you “color” three of the vertices as shown:

[Graph showing colored vertices]

However, B is not bipartite, because it contains a triangular subgraph.
Since B cannot be isomorphic to either of A or C, the answer must be that A and C are isomorphic.