1. (10 pts.) Let \( A = \{1, 3\} \). Suppose that \( f \) is a function from \( A \times A \) to \( \mathbb{Z} \) defined by the rule
\[
f(x, y) = x + y
\]
Write \( f \) as a subset of \( (A \times A) \times \mathbb{Z} \).

Solution:
\[
\begin{align*}
f(1, 1) &= 2 \\
f(1, 3) &= 4 \\
f(3, 1) &= 4 \\
f(3, 3) &= 6
\end{align*}
\]

\( f \) as a subset of \( (A \times A) \times \mathbb{Z} \) is
\[
\{((1, 1), 2), ((1, 3), 4), ((3, 1), 4), ((3, 3), 6)\}
\]
2. For each of the following, determine if the statement is true or false. If true, write a proof. If false, give a counter-example.

(a) (10 pts.) If $a \mid b$ and $b^2 \mid c$, then $a \mid c$.

Proof:

$b = na$ for some $n$.
$c = b^2m$ for some $m$.
$c = (na)^2m = n^2a^2m = a(n^2am)$.
$\therefore, a \mid c$.

(b) (10 pts.) If $a \mid b$ and $b \mid c^2$, then $a \mid c$.

Counter-example:

$a = 4$, $b = 4$ and $c = 2$.

$4 \mid 4$, so $a \mid b$.

$4 \mid 2^2$, so $b \mid c^2$.

However, $4 \nmid 2$. 
3. (10 pts.) Find integers $n$ and $m$ so that $10n + 23m = 1$.

Solution:

$$23 = 2 \cdot 10 + 3$$
$$10 = 3 \cdot 3 + 1$$

$$10 = 3(23 - 2 \cdot 10) + 1$$
$$= 3 \cdot 23 - 6 \cdot 10 + 1$$

$$7 \cdot 10 - 3 \cdot 23 = 1$$

Answer: $n = 7$ and $m = -3$.

4. (10 pts.) Find the unique $x$ with $0 \leq x < 23$ that solves

$$10x \equiv 6 \pmod{23}$$

Solution: (Multiply by 7; reduce mod 23)

$$70x \equiv 42 \pmod{23}$$
$$x \equiv 19 \pmod{23}$$

Answer: $x = 19$.

5. (10 pts.) Find the unique $x$ with $0 \leq x < 230$ that solves

$$x \equiv 1 \pmod{10}$$
$$x \equiv 2 \pmod{23}$$

Solution:

$$x = 2(7 \cdot 10) - 1(3 \cdot 23) \pmod{230}$$
$$= 140 - 69 \pmod{230}$$
$$= 71 \pmod{230}$$

Answer: $x = 71$. 
6. (15 pts.) Use induction to prove that for all \( n \in \mathbb{N} \),
\[
1 + 2 + \cdots + n = \frac{n(n + 1)}{2}
\]

Proof:

Assume \( n = 1 \).

Left Hand Side: \( \frac{1(1 + 1)}{2} = 1 \)

Right Hand Side: \( \frac{n(n + 1)}{2} = 1 \)

Assume that \( n = k + 1 \) with \( k \geq 1 \).

Left Hand Side:
\[
1 + 2 + \cdots + k + (k + 1) = (1 + 2 + \cdots + k) + (k + 1)
= \frac{k(k + 1)}{2} + (k + 1)
= \frac{k^2 + k}{2} + \frac{2k + 2}{2}
= \frac{k^2 + 3k + 2}{2}
\]

Right Hand Side:
\[
\frac{(k + 1)((k + 1) + 1)}{2} = \frac{(k + 1)(k + 2)}{2}
= \frac{k^2 + 3k + 2}{2}
\]
7. (15 pts.) Suppose that a sequence is defined by

\[ \begin{align*}
  a_0 &= 2 \\
  a_1 &= -2 \\
  a_n &= -2a_{n-1} + 3a_{n-2} \quad \text{if} \quad n \geq 2
\end{align*} \]

Prove that \( a_n = (-3)^n + 1 \) for all \( n \in \mathbb{N} \).

Proof:

Assume \( n = 0 \).

\begin{align*}
\text{Left Hand Side:} & \quad a_0 = 2 \\
\text{Right Hand Side:} & \quad (-3)^0 + 1 = 2
\end{align*}

Assume \( n = 1 \).

\begin{align*}
\text{Left Hand Side:} & \quad a_1 = -2 \\
\text{Right Hand Side} & \quad (-3)^1 + 1 = -2
\end{align*}

Assume that \( n = k + 2 \) with \( k \geq 0 \).

\[ a_{k+2} = -2a_{k+1} + 3a_k \]
\[ = -2((-3)^{k+1} + 1) + 3((-3)^k + 1) \]
\[ = -2(-3)^{k+1} - 2 + 3(-3)^k + 3 \]
\[ = -2(-3)^{k+1} - (-3)(-3)^k + 1 \]
\[ = -2(-3)^{k+1} - (-3)^{k+1} + 1 \]
\[ = -3(-3)^{k+1} + 1 \]
\[ = (-3)^{k+2} + 1 \]

Therefore, \( a_n = (-3)^n + 1 \) for all \( n \in \mathbb{N} \).
8. (10 pts.) Solve the recurrence relation.

\[ a_0 = 2 \]
\[ a_1 = 6 \]
\[ a_n = 5a_{n-1} - 6a_{n-2} \quad \text{if} \quad n \geq 2 \]

That is, find a non-recursive formula for the sequence \( a_n \).

Solution:

Step 1: (Characteristic Polynomial)

\[ x^2 = 5x - 6 \]
\[ x^2 - 5x + 6 = 0 \]
\[ (x - 3)(x - 2) = 0 \]
\[ x = 3, 2 \]

Step 2: For some \( c \) and \( d \),

\[ a_n = c3^n + d2^n \]

Step 3: Plug into initial conditions

\[ a_0 = c3^0 + d2^0 = 2 \]
\[ a_1 = c3^1 + d2^1 = 6 \]

\[ c + d = 2 \]
\[ 3c + 2d = 6 \]

\[ -2c + -2d = -4 \]
\[ 3c + 2d = 6 \]

\[ c = 2 \]
\[ d = 0 \]

Answer:

\[ a_n = 2 \cdot 3^n \]