Hw 3 Solutions

Exercise 1(a): Given $A \ast B \ast C$ and $A \ast C \ast D$, prove that $A$, $B$, $C$ and $D$ are distinct points.

Proof: Applying Axiom B1 to $A \ast B \ast C$, we get $A \neq B$, $A \neq C$ and $B \neq C$.
Applying Axiom B1 to $A \ast C \ast D$, we get $A \neq D$ and $C \neq D$.
Finally, suppose that $B = D$.
By substitution in $A \ast C \ast D$, we have $A \ast C \ast B$.
This contradicts $A \ast B \ast C$.
Therefore $B \neq D$.

Proposition 3.5: Given $A \ast B \ast C$. Then $AC = AB \cup BC$ and $B$ is the only point common to segments $AB$ and $BC$.

Choose $P \in AB \cup BC$.
Either $P = A$ or $P = B$ or $P = C$ or $A \ast P \ast B$ or $B \ast P \ast C$.
Case (i): $P = A \implies P \in AC$.
Case (ii): $P = B$ and $A \ast B \ast C \implies A \ast P \ast C \implies P \in AC$.
Case (iii): $P = C \implies P \in AC$.
Case (iv): $A \ast P \ast B$ and $A \ast B \ast C \implies A \ast P \ast C \implies P \in AC$.
Case (v): $B \ast P \ast C$ and $A \ast B \ast C \implies A \ast P \ast C \implies P \in AC$.
Therefore $AB \cup BC \subset AC$.

Now choose $Q \in AC$.
Either $Q = A$ or $Q = C$ or $A \ast Q \ast C$.
Case (i): $Q = A \implies Q \in AB$.
Case (ii): $Q = C \implies Q \in BC$.
Case (iii): $A \ast Q \ast C$.

We know $Q \neq A$, so we have the following subcases:
Subcase (a): $Q = B \implies Q \in AB$.
Subcase (b): $A \ast Q \ast B \implies Q \in AB$.
Subcase (c): $A \ast B \ast Q$ and $A \ast Q \ast C \implies B \ast Q \ast C$.
Therefore $Q \in BC$.
Subcase (d): $B \ast A \ast Q$ and $A \ast Q \ast C \implies B \ast A \ast C$,
which contradicts $A \ast B \ast C$.
In all cases we have $Q \in AB$ or $Q \in BC$.
Therefore $AC \subset AB \cup AC$.
Proof: Part 2.

Clearly $B$ is common to both $AB$ and $BC$.
Suppose that $P \neq B$ is a point common to both $AB$ and $BC$.
We now have $P = A$ or $A \ast P \ast B$.
We also have $P = C$ or $B \ast P \ast C$.
There are four cases.
Case (i): $P = A$ and $P = C$.
   This implies $A = C$, contradicting $A \ast B \ast C$.
Case (ii): $P = A$ and $B \ast P \ast C$.
   This implies $B \ast A \ast C$, contradicting $A \ast B \ast C$.
Case (iii): $A \ast P \ast B$ and $P = C$.
   This implies $A \ast C \ast B$, contradicting $A \ast B \ast C$.
Case (iv): $A \ast P \ast B$ and $B \ast P \ast C$.
   $B \ast P \ast C$ and $A \ast B \ast C \implies A \ast B \ast P$, contradicting
   $A \ast P \ast B$.
Therefore there is no point other than $B$ common to $AB$ and $BC$. 