Exam 2 Key

Problem 1. Prove that an equilateral triangle has all angles congruent.

Proof: Suppose that \( \triangle ABC \) is equilateral.

\[
AB \cong AC.
\]
\[
BC \cong CB.
\]
\[
AC \cong AB.
\]
By SSS, \( \triangle ABC \cong \triangle ACB \).
\[
\therefore \angle B \cong \angle C.
\]
Similarly, \( \triangle ABC \cong \triangle CBA \).
\[
\therefore \angle A \cong \angle C.
\]
It follows that all three angles are congruent \( \square \)

Problem 2. Given \( \triangle ABC \), prove that \( AC > BC \) implies \( \angle A < \angle B \).

Proof: Since \( AC > BC \), there is a point \( E \) such that \( A \neq E \neq C \) and \( BC \cong EC \).

By the crossbar theorem, \( BE \) is inside \( \angle ABC \).
\[
\therefore \angle CBE < \angle CBA.
\]
In \( \triangle CBE \), \( \angle CBE \cong \angle CEB \).
By the exterior angle theorem, \( \angle A < \angle CEB \).
\[
\therefore \angle A < \angle CBA \square
\]

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**Problem 3.** Suppose \( \triangle ABC \) is isosceles with \( AB \cong CB \). If \( D \) is on \( AC \) as shown, prove that \( BD \) bisects \(<ABC \) if and only if \( D \) is the midpoint of \( AC \).

**Proof:** First suppose that \( D \) is the midpoint of \( AC \).

This means \( AD \cong CD \).

We are given \( AB \cong CB \).

\( BD \cong BD \).

By SSS, \( \triangle ABD \cong \triangle CBD \).

\( \therefore \angle ABD \cong \angle CBD \).

Thus \( BD \) bisects \( \angle ABC \).

Now assume \( BD \) bisects \( \angle ABC \).

By definition of bisector, \( \angle ABD \cong \angle CBD \).

We are given \( AB \cong CB \).

\( BD \cong BD \).

By SAS, \( \triangle ABD \cong \triangle CBD \).

\( \therefore AD \cong CD \).

Thus \( D \) is the midpoint of \( AC \) ■

**Problem 4.** Prove that \( \triangle ABC \) has a unique altitude from \( A \).

**Proof:** Let \( l \) be a line perpendicular to \( BC \) through \( A \).

Let \( D \) be the point where \( l \) meets \( BC \).

\( AD \) is an altitude from \( A \).

Now let \( AE \) be another altitude from \( A \).

Suppose that \( E \neq D \).

\( AE \) and \( AD \) both meet \( BC \) in right angles.

By alternate interior angles, \( \overrightarrow{AE} \parallel \overrightarrow{AD} \).

However, these lines meet at \( A \) ■
Problem 5. Given $ABCD$ with $AC \perp AB$, $BD \perp AB$, and $AC \cong BD$. Prove that Hilbert’s parallel postulate implies $ABCD$ is a rectangle.

Proof: By alternate interior angles, $\overrightarrow{AC} \parallel \overrightarrow{BD}$.

Consider the transversal $BC$.
Hilbert’s parallel postulate implies the converse alternate interior angle theorem.
\[ \therefore \angle ACB \cong \angle DBC. \]
$CB \cong BC$.
$AC \cong DB$.
By SAS, $\triangle ACB \cong \triangle DBC$.
\[ \therefore \angle A \cong \angle D. \]
This means $\angle D$ is a right angle.
A similar argument using $\overrightarrow{AD}$ proves $\angle C \cong \angle B$.
Therefore $ABCD$ is a rectangle. ■